Proof of Lemma 1.

Let u_θ be a unit vector along $a_j a_k$ (and assume that z is a unit vector). Look at

$$\mu(a_k) = \mu(tz + ru_\theta) = e^{-\frac{d^2}{2}} e^{-\frac{(s+r)^2}{2}},$$

where d and s are defined at figure 2.

Proposition 1. *For $s > 1$, $r \le \frac{1}{s}$*

$$\frac{e^{-\frac{s^2}{2}}}{e^{-\frac{(s+r)^2}{2}}} \le e^2$$

As a corollary, for $0 < r_1 < r_2 < \frac{1}{\sqrt{8 \lg n}}$ holds

$$\frac{\mu(tz + r_1 u_\theta)(l + r_1)}{\mu(tz + r_2 u_\theta)(l + r_2)} \le e^2.$$

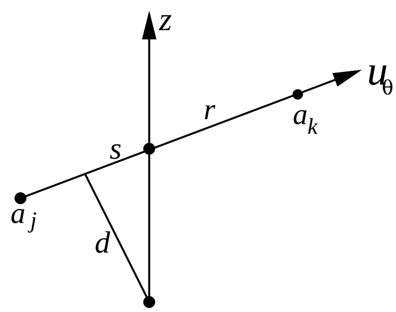

Figure 2:

Proposition 2. *Let f be s.t. for $0 < x_1 < x_2 < K$ and $\frac{f(x_1)}{f(x_2)} < c$. Then*

$$\frac{\int_0^\epsilon f(x)\,dx}{\int_0^K f(x)\,dx} \leq \frac{\epsilon c}{K}.$$

Proof. One can split the interval $[0, K]$ into $\frac{K}{\epsilon}$ subintervals of length ϵ. The integral of f on each subinterval is lower bounded by $c^{-1}\int_0^\epsilon f(x)\,dx$, thus $\int_0^K f(x)\,dx \geq \frac{K}{\epsilon}c^{-1}\int_0^\epsilon f(x)$. □

It is sufficient to choose $K = 1/\sqrt{8\lg n}$ to finish the proof of Lemma 1.■

Proof of Lemma 2.

Let

$$g(\theta) = \underbrace{\left(\int_a [CH_{j,k}] \prod \mu(a_i)\right)}_{g_1(\theta)} \sin(\theta) \underbrace{\mu(tz + ru_\theta)\mu(tz - lu_\theta)}_{g_2(\theta)}$$

In order to estimate the ratio (2) it will be sufficient to confine ourselves to $0 < \theta < \frac{1}{16\lg n}$ in the denominator. For $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ holds

$$\frac{g_1(\theta_1)}{g_1(\theta_2)} \leq 1.$$

To see this, notice that $g_1(\theta)$ is the probability of the rest of the points a_i lying on the origin side of the line $a_j a_k$. This probability decreases when the distance of the line $a_j a_k$ to $\bar{o}$ decreases, thus $g_1(\theta)$ is monotone.

As in Proposition 1, for $t < \sqrt{8 \lg n}$; $l, r < 2\sqrt{8 \lg n}$; $0 < \theta_1 < \theta_2 < \frac{1}{16 \lg n}$ holds

$$\frac{\mu(tz + ru_{\theta_1})}{\mu(tz + ru_{\theta_2})} < e^2,$$

which implies for $0 < \theta_1 < \theta_2 < \frac{1}{16 \lg n}$

$$\frac{g_2(\theta_1)}{g_2(\theta_2)} \leq e^2.$$

Finally, for small values of θ, $\sin(\theta) \sim \theta$, so for $0 < \theta_1 < \theta_2 < \frac{1}{16 \lg n}$

$$\frac{g(\theta_1)}{g(\theta_2)} \leq 2e^4 \frac{\theta_1}{\theta_2}.$$

The following fact is the analog of Proposition 2:

Proposition 3. *If for* $x_1 < x_2$ $\frac{f(x_1)}{f(x_2)} \leq c\frac{x_1}{x_2}$ *then*

$$\frac{\int\limits_0^{\epsilon} f(x)\,dx}{\int\limits_0^{K} f(x)\,dx} \leq 4c\left(\frac{\epsilon}{K}\right)^2$$

It is left to set $K = \frac{1}{16 \lg n}$. The lemma is proven.■

At the end we justify the change of variables $(a_j, a_k) \to (l, r, t, \theta)$ that we made in the proof and compute the Jacobian of this transform. Let $a = a_j$ and $b = a_k$ be two points in $\mathbf{R}^2$, specified by four parameters l, r, h, θ as shown on figure 1. By the straightforward calculation,

$a_x = l \sin(\theta)$

$a_y = t - l \cos(\theta)$

$b_x = r \sin(\theta)$

$b_y = t + r \cos(\theta)$

The Jacobi matrix

$$
J = \begin{array}{cc}
\begin{array}{cccc} \partial a_x & \partial a_y & \partial b_x & \partial b_y \end{array} & \\
\begin{pmatrix}
0 & 0 & \sin(\theta) & \cos(\theta) \\
\sin(\theta) & -\cos(\theta) & 0 & 0 \\
l\cos(\theta) & l\sin(\theta) & r\cos(\theta) & -r\sin(\theta) \\
0 & 1 & 0 & 1
\end{pmatrix} & \begin{array}{c} \partial r \\ \partial l \\ \partial \theta \\ \partial t \end{array}
\end{array}
$$

and the Jacobian

$$
|\det J| = (l + r)\sin(\theta),
$$

hence

$$
da\, db = (l + r)\sin(\theta)\, dr\, dl\, d\theta\, dt\,.
$$