Problem Set Number 1, 18.385j/2.036j
MIT (Fall 2014)

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1   Get equation from phase line portrait problem #07.

Statement: Get equation from phase line portrait problem #07.

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \tag{1.0.1}$$

where f is some function which is (at least) Lipschitz continuous. Assume that (1.0.1) has at least two critical points, and let $x_1 < x_2$ be two consecutive critical points (there is no other critical point between them). Assume now that both x_1 and x_2 are stable. **Is this possible? Does a function $f = f(x)$ yielding this exist?**

If the answer is no, prove it.

If the answer is yes, prove it by giving an example.

2 Get equation from phase line portrait problem #08.

Statement: Get equation from phase line portrait problem #08.

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \tag{2.0.1}$$

where f is some function which is (at least) Lipschitz continuous. Assume that (2.0.1) has infinitely many critical points $-\infty < \ldots x_n < x_{n+1} < \ldots \infty$. Assume that all the critical points are semi-stable.[1] **Is this possible? Does a function $f = f(x)$ yielding this exist?**

If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

3 Problem 02.02.10 - Strogatz (Fixed points).

Statement for problem 02.02.10.

(Fixed points). For each of (A)–(E) below, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. In all cases assume that $f(x)$ is a smooth function.

  A. Every real number is a fixed point.

  B. Every integer is a fixed point, and there are no others.

  C. There are precisely three fixed points, and all of them are stable.

  D. There are no fixed points.

  E. There are precisely 100 fixed points.

4 Problem 02.03.04 - Strogatz (The Alle effect).

Statement for problem 02.03.04.

(The Allee effect). For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N. This is called the Allee effect.[2] For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

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[1] A critical point is semi-stable if the solutions diverge from the critical point on one side, and converge on the other.
[2] Edelstein-Keshet, L. (1988) *Mathematical Models in Biology* (Random House, NY).

  a. Show that $\dot{N}/N = r - a(N - b)^2$ provides an example of the Allee effect, if r, a, and b satisfy certain constraints, to be determined.

  b. Find all the fixed points of the system, and classify their stability.

  c. Sketch the solution $N(t)$ for different initial conditions.

  d. Compare the solutions $N(t)$ with those found for the logistic equation. What are the qualitative and or quantitative differences, if any?

*Hint: for parts **c** and **d** write the equation in terms of the a-dimensional variables $x = N/b$ and $\tau = (a\,b^2)\,t$. The equation will then have a single parameter λ — the growth rate for $x \approx 1$. Find λ and show that it satisfies a constraint $\lambda > \lambda_c$. Investigate then the behavior of the solutions of biological interest for various values of λ. In particular: what happens for λ close to λ_c?*

5 Problem 02.05.05 - Strogatz (Non-uniqueness example).

Statement for problem 02.05.05.

(A general example of non-uniqueness). Consider the initial value problem $\dot{x} = |x|^{p/q}$, $x(0) = 0$, where p and q are positive integers with no common factors.
a. Show that there are an infinite number of solutions if $p < q$.
b. Show that there is a unique solution if $p > q$.
Hint: Use Lipschitz continuity to show uniqueness.

6 Problem 02.05.06 - Strogatz (The leaky bucket).

Statement for problem 02.05.06.

(The leaky bucket). The following example[3] shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see a water bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying[4] a minute ago,

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[3]Hubbard, J. H., and West, B. H. (1991) *Differential Equations: A Dynamical Systems Approach, Part I* (Springer, New York).

[4]Note that, in this problem, evaporation effects are neglected.

ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Here is a crude model for the situation. Let $h(t)$ = height of the water remaining in the bucket at time t; a = area of the hole; A = cross-sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

a. Show that $a\,v(t) = A\,\dot{h}$. What physical law are you invoking? **Warning:** since $\dot{h} < 0$, this presumes that *we assign a negative value to the velocity v*. This is a weird choice, implicit in the problem statement, but acceptable.

b. To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount Δh, and that the water has density ρ. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation $v^2 = 2\,g\,h$ — g = gravity acceleration.

c. Combining **a** and **b**, show that $\dot{h} = -C\,\sqrt{h}$, where $C = \dfrac{a}{A}\,\sqrt{2\,g}$.

d. Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique *in backwards time*, i.e., for $t < 0$.

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7   Problem 03.02.06 - Strogatz (Eliminate the cubic term).

Statement for problem 03.02.06.

(Eliminating the cubic term). Consider the system

$$\frac{dX}{dt} = RX - X^2 + aX^3 + O(X^4), \tag{7.0.1}$$

where $R \neq 0$. We want to find a new variable x such that the system transforms into

$$\frac{dx}{dt} = Rx - x^2 + O(x^4). \tag{7.0.2}$$

This would be a big improvement, since the cubic term has been eliminated and the error term has been bumped to fourth order. In fact, the procedure to do this (sketched below) can be generalized to higher orders.[5] This generalization is the subject matter of problem 03.02.07.

Let $\boxed{x = X + bX^3 + O(X^4),}$ where b is chosen later to eliminate the cubic term in the differential equation for x. This is called a **near-identity transformation**, since x and X are practically equal: they differ by a cubic term.[6] Now we need to rewrite the system in terms of x; this calculation requires a few steps.

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[5] That is, one can successively eliminate all the higher order terms: $O(x^3)$, $O(x^4)$, ..., etc.

[6] We have skipped the quadratic term X^2, because it is not needed — you should check this later.

1. Show that the near-identity transformation can be inverted to yield $\boxed{X = x + cx^3 + O(x^4),}$ and solve for c.

2. Write $\boxed{\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4),}$ and substitute for X and $\dot{X}$ on the right hand side, so that everything depends only on x. Multiply the resulting series expansions and collect terms, to obtain $\boxed{\dot{x} = Rx - x^2 + kx^3 + O(x^4),}$ where k depends on a, b, and R.

3. Now the moment of triumph: choose b so that $k = 0$.

4. Is it really necessary to make the assumption that $R \neq 0$? Explain.

8   Problem 03.04.05 - Strogatz (Find and classify bifurcations).

Statement for problem 03.04.05.

For equation (8.0.1) below, find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$\frac{dx}{dt} = r - 3x^2 .\tag{8.0.1}$$

9   Baby normal forms

Do problems #2 and #3 in the *Bifurcations: baby normal forms* notes in the lecture notes. Note that, *in problem #3, what you are expected to do is to answer the "question" at the very end — in large **boldface**.*

THE END.

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