Lecture 15: Non-identically Distributed Steps and Random Waiting Times

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In this lecture we conclude our discussion on non-identically distributed steps and introduce random waiting times between subsequent steps.

1 Non-Identically Distributed Steps - Decaying Bernoulli Walk

As we have seen from last lecture, the position of the simple (Bernoulli) random walker with decaying non-identical steps of the form $\Delta x_m = a^m \varepsilon_m$, is given by:

$$X_N = \sum_{m=1}^{N} a^m \varepsilon_m \tag{1}$$

where $a < 1$ and ε_m takes the values $\pm 1$with equal probability $1/2$. Recall that the changes in the stepsize a can dramatically change the shape of the distribution. In particular, we provided a qualitative description for the following cases:

- $a = 1/3$: X_N is uniformly distributed on the Cantor middle thirds set (see figure 1).
- $a = 1/2$: X_N is uniform on the interval $[-1, 1]$.
- $a < 1/2$: X_N is uniformly distributed on fractals, i.e. the random walk is non recurrent, which means that the walker never returns to a previous location.

Here we present a more quantitative analysis, using the formalism we developed from previous lectures. Letting p_m be the PDF for the Δx_m we have:

$$p_m(x; a) = \frac{1}{2}\left[\delta(x + a^m) + \delta(x - a^m)\right] \Longrightarrow \widehat{p_m}(k; a) = \cos(ka^m) \tag{2}$$

The PDF for X_N, P_N, is a convolution of the $\{\widehat{p_m}\}$ and becomes a product in the Fourier space:

$$\widehat{P_N}(k; a) = \prod_{m=1}^{N} \widehat{p_m}(k; a) = \prod_{m=1}^{N} \cos(ka^m) \tag{3}$$

This product is in general hard to evaluate but when $a = 1/2$ we get:

$$\widehat{P_N}(k; 1/2) = \cos(k/2) \cdot \cos(k/2^2) \ldots \cos(k/2^N) = \frac{\sin(k)}{2\sin(k/2)} \frac{\sin(k/2)}{2\sin(k/4)} \cdots \frac{\sin(k/2^{N-1})}{2\sin(k/2^N)} = \frac{\sin(k)}{2^N \sin(k/2^N)}$$

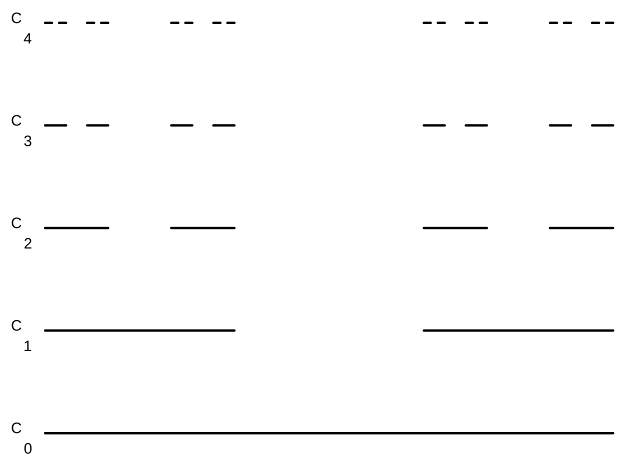

Figure 1: The first few stages $C_0, C_1, C_2, \ldots$ in the construction of the middle-thirds Cantor set

In the limit $N \to \infty$ we have $\widehat{P_\infty}(k; 1/2) = \frac{\sin k}{k}$. The inverse Fourier Transform of $\widehat{P_\infty}(k; 1/2)$ is the uniform distribution on $[-1, 1]$:

$$P_\infty(x; 1/2) = \begin{cases} 1/2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases} \tag{4}$$

One can get more insight on the limiting PDFs for other values of a by using the properties of the product and regrouping terms:

$$\begin{aligned} \widehat{P_\infty}(k; a) &= \left(\prod_{m=1}^{\infty} \cos\left(ka^{2m+1}\right) \right) \left(\prod_{n=1}^{\infty} \cos\left(ka^{2n}\right) \right) \\ &= \widehat{P_\infty}\left(\frac{k}{a}; a^2\right) \cdot \widehat{P_\infty}\left(k; a^2\right) \\ &= \widehat{P_\infty}\left(\frac{k}{a^2}; a^3\right) \cdot \widehat{P_\infty}\left(\frac{k}{a}; a^3\right) \cdot \widehat{P_\infty}\left(k; a^3\right) \\ &\vdots \end{aligned}$$

Since $P_\infty(x; 1/2)$ is known, we can get $P_\infty\left(x; 1/2^{1/m}\right)$ for $m = 1, 2, \ldots$. When $a = 1/\sqrt{2}$:

$$P_\infty\left(x; \frac{1}{\sqrt{2}}\right) = P_\infty\left(\frac{x}{\sqrt{2}}; \frac{1}{2}\right) * P_\infty\left(x; \frac{1}{2}\right) \tag{5}$$

Hence $P_\infty\left(x; 1/\sqrt{2}\right)$ is given by the convolution of two uniform distributions on the intervals $[-1, 1]$ and $\left[-\sqrt{2}, \sqrt{2}\right]$. We can go on with this procedure to get (see figure 2)

$$P_\infty\left(x; \frac{1}{2^{1/4}}\right) = P_\infty\left(\frac{x}{2^{1/4}}; \frac{1}{\sqrt{2}}\right) * P_\infty\left(x; \frac{1}{\sqrt{2}}\right) \tag{6}$$

Also note that we can write $\widehat{P_N}(k; a) = \widehat{P_N}\left(ka^{N+1}; \frac{1}{a}\right)$ which implies that a and $1/a$ show an equivalent limiting distribution with $< X_\infty^2 > = \frac{a^2}{1-a^2}$.

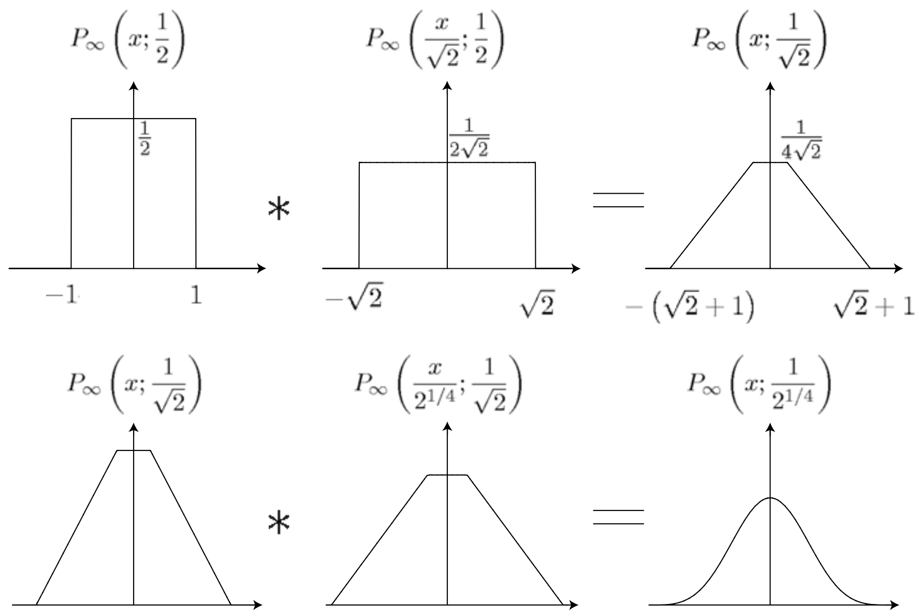

Figure 2: Using convolutions to obtain $P_\infty\left(x;1/2^{1/m}\right)$

2 Random Waiting Times

Thus far, we have considered random walks separated with equal time steps, τ. Here we introduce random waiting times between steps. In particular, τ_n is assumed to be an iid random variable with PDF $\psi(t)$, called the waiting time density. The analysis for such walks is very much like adding displacements, which are now positive. This slight difference suggests that we use the Laplace transform, defined as:

$$\tilde{\psi}\left(s\right)=\int_0^\infty e^{-st}\psi\left(t\right)dt \tag{7}$$

If this integral is convergent for $s=s_0$ then it converges for all $\Re e\{s\}>\Re e\{s_0\}$, called the half-plane of convergence (see figure 3). The inverse Laplace transform, also called the Bromwich integral is given by:

$$\psi\left(t\right)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\tilde{\psi}\left(s\right)ds \tag{8}$$

where γ can take any value that lies in the half-plane of convergence; thus the path of integration can be any vertical line to the right of all singularities of $\tilde{\psi}\left(s\right)$. As with Fourier transforms, one may be also interested in the $\ell-$th moment which is easily expressed as:

$$m_\ell=\left\langle\tau^\ell\right\rangle=\int_0^\infty t^\ell\psi\left(t\right)dt=\left(-\frac{d}{ds}\right)^\ell\tilde{\psi}\left(s\right)\Big|_{s=0} \tag{9}$$

Obviously $m_0=\tilde{\psi}\left(0\right)=1$ and we can also expand $\tilde{\psi}$ in terms of its moments:

$$\tilde{\psi}\left(s\right)=\sum_{\ell=0}^{\infty}\frac{\left(-s\right)^\ell\left\langle\tau^\ell\right\rangle}{\ell!}=1-\left\langle\tau\right\rangle s+\frac{s\left\langle\tau^2\right\rangle}{2}-... \tag{10}$$

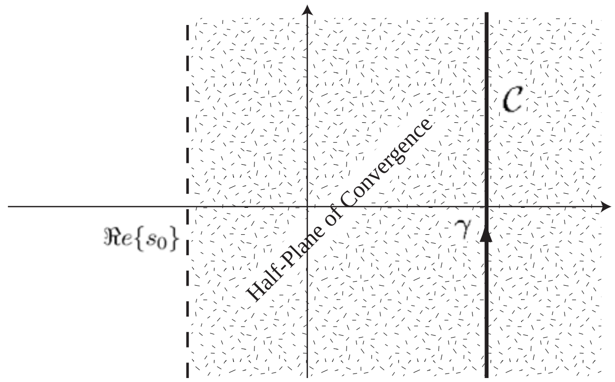

Figure 3: The half-plane of convergence for the Laplace transform

If we add two random variables τ_1 and τ_2, which are iid waiting times with PDF $\psi(t)$, so that $t = \tau_1 + \tau_2$ we have:

$$\psi_2(t) = \int_0^t \psi(t')\,\psi(t-t')\,dt' = \psi^{*2}(t) \tag{11}$$

where we denote with $\psi_N(t)$, the PDF for time t at the $N-$th step and in general, $\psi_N(t) = \psi^{*N}(t)$. As usual, taking the Laplace transform of the convolutions we get the product $\tilde{\psi}_N(s) = \left[\tilde{\psi}(s)\right]^N$.

2.1 Finite Mean Waiting Times

It is worthwhile to note that if a finite mean waiting time exists, $\bar{\tau}$, then in the long time limit, which implies $s \to 0$, we may expand $\tilde{\psi}_N(s)$ as:

$$\tilde{\psi}(s) \sim 1 - s\bar{\tau} \Rightarrow \tilde{\psi}_N(s) \sim \lim_{N\to\infty} (1 - s\bar{\tau})^N \Rightarrow \tilde{\psi}_N(s) \sim e^{-Ns\bar{\tau}} \tag{12}$$

This last expression is valid provided that Ns remains fixed. Inverting the transform we get:

$$\psi_N(t) \sim \delta(t - N\bar{\tau}) \tag{13}$$

So the random variable $z = \frac{t_N}{N\bar{\tau}}$ tends to the limiting PDF $\delta(z-1)$. Hence we may conclude that a random walk with finite mean waiting times is equivalent to a discrete random walk (up to leading order statistics), with the walker moving with that mean waiting time $\bar{\tau}$. Depending on whether the variance σ_τ is finite or not we can have:

- If $\sigma_\tau < \infty$, one may easily deduce that $\psi_N(t) \sim \frac{1}{\sqrt{2\pi\sigma_\tau^2 N}} e^{-\frac{(t-N\bar{\tau})^2}{2\sigma_\tau^2 N}}$, in accordance with the Central Limit Theorem.
- If $\sigma_\tau = \infty$, then we may expand $\tilde{\psi}(s) \sim 1 - s\bar{\tau} + a\,s^\alpha$ for some $\alpha \in (1,2)$. In this case $\psi_N(t)$ tends to the asymmetric Lévy distribution $\psi_N(t) \sim \ell_{\alpha,1}\left((t - N\bar{\tau})/N^{1/\alpha}\right)$.

In either case the width of the distribution is much smaller than $N\bar{\tau}$, so we can say that $t/N\bar{\tau}$ tends to 1 in probability.

2.2 Infinite Mean Waiting Times

Now we examine the case $\bar{\tau} = \infty$. After properly re-normalizing, $\psi_N(t)$ tends to the one sided Lévy stable law :

$$\tilde{\psi}(s) = \widetilde{\ell_{\alpha,1}} = e^{-b\,s^{\alpha}} \overset{Laplace}{\Longrightarrow} \psi(t) = \ell_{\alpha,1}(t;b) \tag{14}$$

This is consistent with the Fourier transform representation derived in Lecture 13. To illustrate this point, start with the inverse Laplace transform for $\tilde{\psi}(s)$:

$$\psi(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st-bs^{\alpha}}\,ds \tag{15}$$

Making the change of variable $s = ik$ and noting that:

$$\begin{aligned}(ik)^{\alpha} &= \begin{cases} e^{\frac{\pi i\alpha}{2}}k^{\alpha} & \text{for } k>0 \\ e^{\frac{-\pi i\alpha}{2}}(-k)^{\alpha} & \text{for } k<0\end{cases} \\ &= |k|^{\alpha}\left(\cos\frac{\pi\alpha}{2} - i\,\mathrm{sgn}(k)\sin\frac{\pi\alpha}{2}\right)\end{aligned} \tag{16}$$

we may rewrite the integral as:

$$\psi(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikt-b|k|^{\alpha}\left(\cos\frac{\pi\alpha}{2}-i\mathrm{sgn}(k)\sin\frac{\pi\alpha}{2}\right)}dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikt-b\cos\frac{\pi\alpha}{2}|k|^{\alpha}\left(1-i\mathrm{sgn}(k)\tan\frac{\pi\alpha}{2}\right)}dk \tag{17}$$

and we can conclude that:

$$\widehat{\ell_{\alpha,1}} = e^{-\tilde{b}|k|^{\alpha}\left(1-i\mathrm{sgn}(k)\tan\frac{\pi\alpha}{2}\right)} \tag{18}$$

where $\tilde{b} = b\cos\frac{\pi\alpha}{2}$, thus obtaining the result we got in Lecture 13 for the case when $\alpha < 1$ and $\beta = 1$.

2.2.1 Smirnov Density

In general, integrals of the type we have seen in the previous section are hard to express in terms of elementary functions. Here we present an example where we can actually evaluate that integral. In particular, we consider the case where $\alpha = 1/2$ so that we have to compute the integral:

$$\ell_{1/2,1}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st-b\sqrt{s}}ds \tag{19}$$

The square root in the integrand suggests that we introduce a branch cut along the negative t-axis and deform our original contour of integration ($\mathcal{C}$) to a contour ($\mathcal{C}'$) which wraps around the branch cut (see figure 4). Then our original integral becomes:

$$\ell_{1/2,1}(t) = \frac{1}{2\pi i}\left(\int_{-\infty}^{0} e^{st+ib\sqrt{-s}}ds + \int_{0}^{-\infty} e^{st-ib\sqrt{-s}}ds\right) \tag{20}$$

A change in variable $s = -\xi$ gives:

$$\ell_{1/2,1}(t) = \frac{1}{2\pi i}\left(\int_{+\infty}^{0} -e^{-\xi t+i\sqrt{\xi}b}d\xi + \int_{0}^{+\infty} -e^{-\xi t-i\sqrt{\xi}b}d\xi\right) = \frac{1}{\pi}\int_{0}^{+\infty} e^{-\xi t}\sin b\sqrt{\xi}\,d\xi \tag{21}$$

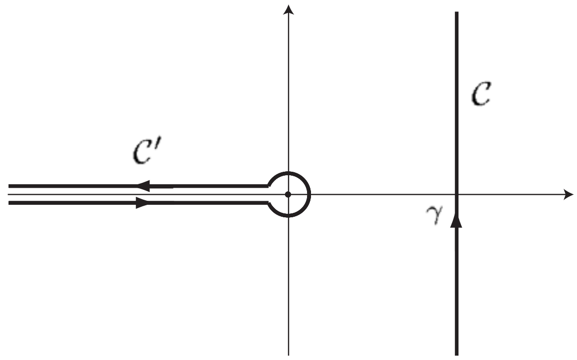

Figure 4: Contour deformation for $\ell_{1/2,1}$

Using the Taylor expansion for sine we get:

$$\ell_{1/2,1}(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-\xi t}\sum_{n=0}^{\infty}\frac{(-1)^n\left(b\sqrt{\xi}\right)^{2n+1}}{(2n+1)!}d\xi = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n b^{2n+1}}{(2n+1)!}\int_0^{+\infty} e^{-\xi t}\xi^{n+\frac{1}{2}}d\xi \tag{22}$$

With another change of variable $\eta = \xi t$ we finally obtain:

$$\ell_{1/2,1}(t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n b^{2n+1}}{(2n+1)!}\frac{1}{t^{n+3/2}}\int_0^{+\infty} e^{-\eta}\eta^{n+\frac{1}{2}}d\eta = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n b^{2n+1}}{(2n+1)!}\frac{1}{t^{n+3/2}}\Gamma\left(n+\frac{3}{2}\right) \tag{23}$$

We can simplify this further by using the difference relation $\Gamma(x) = (x-1)\Gamma(x-1)$:

$$\Gamma\left(n+\frac{3}{2}\right) = \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)...\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{1}{2^n}\left[(2n+1)(2n-1)...1\right]\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2^n}\frac{(2n+1)!}{2n\times 2(n-1)\times ...2} = \frac{(2n+1)!}{2^{2n}n!}\Gamma\left(\frac{1}{2}\right) = \frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}n!} \tag{24}$$

Putting everything together and noting that the summation gives the exponential function, we finally obtain:

$$\ell_{1/2,1}(t) = \frac{b}{2\sqrt{\pi}t^{3/2}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{b^2}{4t}\right)^n \Rightarrow \ell_{1/2,1}(t) = \sqrt{\frac{b^2}{4\pi t^3}}\exp\left(-\frac{b^2}{4t}\right) \tag{25}$$

A plot of this function is shown in figure 5. It has $< t >= \infty$ and an essential singularity at 0. As we can see it increases rapidly at the beginning and then it falls like $\frac{1}{t^{1+\alpha}} = \frac{1}{t^{3/2}}$ (dashed line in figure).

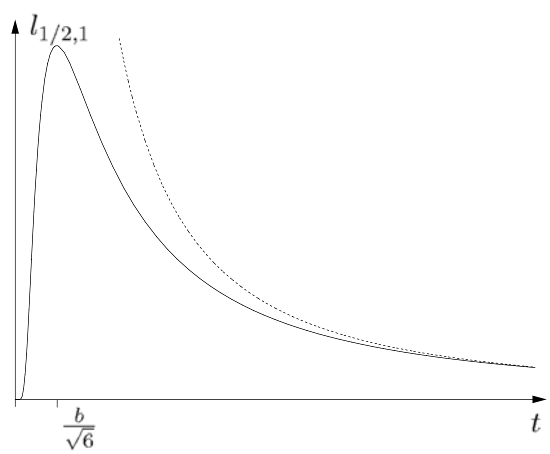

Figure 5: Smirnov density