18.366 Random Walks and Diffusion, Spring 2005, M. Z. Bazant.

Exam 1

Due at 9:30am in lecture on Thursday March 10.

Directions: (i) Work independently and not discuss the exam with anyone. (ii) You may quote formulae from 2005 18.366 lecture notes and problem sets, but otherwise all steps in your solutions should be derived in detail. (iii) You may consult the required and recommended books, but no other resources (books, web sites, etc.).

1. **Multivariate normal random walk.** Consider a random walk of N independent (but *not* identically distributed) steps in d dimensions

$$\vec{X}_N = \sum_{n=1}^{N} \vec{x}_n$$

where the PDF of the nth step, $\vec{x}_n$, is

$$p_n(\vec{x}) = \frac{\exp\left(-\frac{1}{2}\vec{x}\cdot C_n^{-1}\vec{x}\right)}{(2\pi)^{d/2}|C_n|^{1/2}}$$

and C_n is the second-moment matrix, $(C_n)_{ij} = \langle x_n^{(i)} x_n^{(j)}\rangle$. Find the PDF, $P_N(\vec{x})$, of $\vec{X}_N$.

2. **Student random walk.** Consider a random walk with N IID steps sampled from the following PDF (a special case of the "student distribution"):

$$p(x) = \frac{A}{(1+x^2)^2}.$$

(a) Derive the characteristic function, $\hat{p}(k)$, and obtain the tail amplitude, A, and variance, σ^2.

(b) Find the leading asymptotic behavior of the PDF, $P_N(x)$, of the position, X_N, in the tail, $x \to \infty$.

(c) Let $\phi_N(z)$ be the PDF of $Z_N = X_N/\sigma\sqrt{N}$. Derive the first two terms in the asymptotic expansion

$$\phi_N(z) \sim \phi(z) + \frac{g_1(z)}{\sqrt{N}}$$

as $N \to \infty$ with $z = O(1)$ (inside the "central region" of $P_N(x)$).

(d) Now derive at least *four* terms in the expansion:

$$\phi_N(z) \sim \phi(z) + \frac{g_1(z)}{N^{1/2}} + \frac{g_2(z)}{N} + \frac{g_3(z)}{N^{3/2}} + \dots .$$

3. **The largest step.** Consider a random walk with N IID steps $\{x_n\}$ sampled from a PDF, $p(x)$, with $P(x) = \text{Prob.}(x_n < x) = \int_{-\infty}^{x} p(y)dy$. Using the notation of order statistics, consider the largest step:

$$x_{(N)} = \max_{1 \le n \le N} x_n.$$

   (a) Show that the CDF of $x_{(N)}$ is given by

   $$F_N(x) = \text{Prob.}(x_{(N)} < x) = P(x)^N.$$

   (b) Find an equation for $x_{max}(N)$, the most probable value of $x_{(N)}$.

   (c) Consider a step PDF with a power-law tail, $p(x) \sim A/x^{1+\alpha}$ as $x \to \infty$, and find the asymptotic form of $x_{max}(N)$ as $N \to \infty$. Compare the scaling of $x_{max}(N)$ with the width of the "central region", $x_c(N) = \sigma\sqrt{N}$ for $\alpha > 2$.

   (d) Based on (c), explain why the scaling of the random walk must change for $0 < \alpha \le 2$. Assuming the largest step dominates the position for $0 < \alpha < 2$, derive the "anomalous" scaling exponent $\nu(\alpha)$ of the width of the distribution, $x_w(N) \propto N^\nu$. (By "width", we mean that $Z_N = X_N/x_w(N)$ has a well-defined limiting PDF.) Compare with the exact result for the Cauchy random walk ($\alpha = 1$) from Problem Set 1.