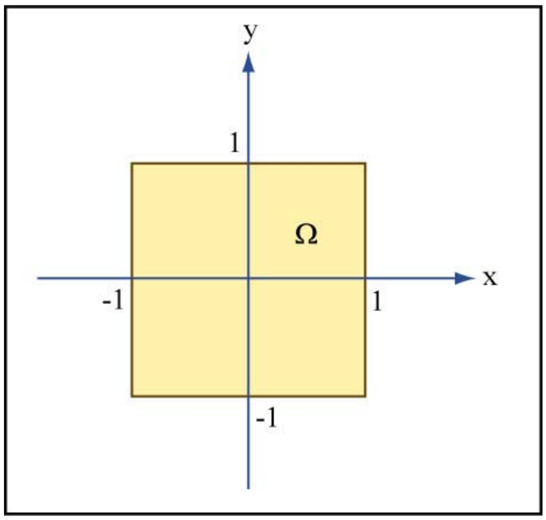
Elliptic Equations and Linear Systems

In rectangular geometries, construct 2D/3D from 1D by Tensor product.

Ex.: $\left\{ \begin{array}{rl} -(u_{xx}+u_{yy}) = f & (x,y) \in]-1,1[^2 \\ u = 0 & \text{on } \partial\Omega \end{array} \right\}$



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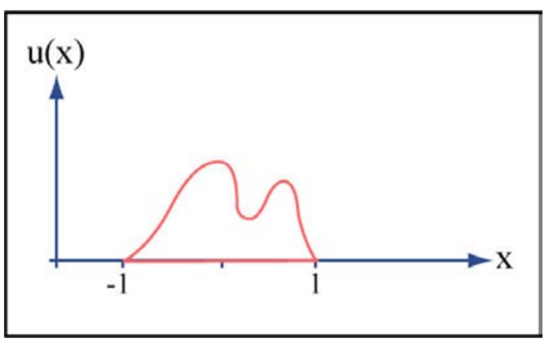
Assume have 1D discretization of

$\left\{ \begin{array}{rl} -u_{xx} = f &]-1,1[\\ u = 0 & \{-1,1\} \end{array} \right\}$



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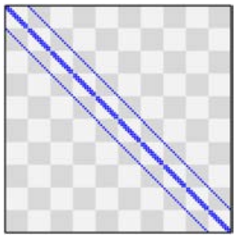
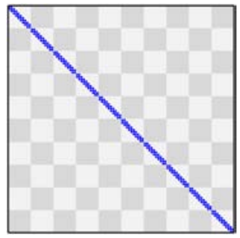
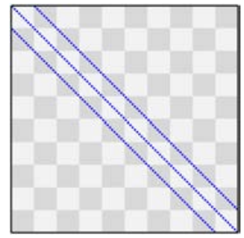
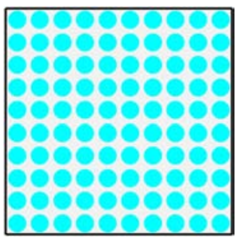
e.g.: • finite difference

$K_{1D} =$  sparse

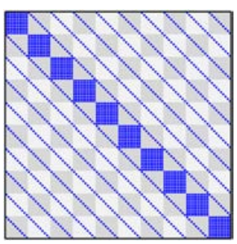
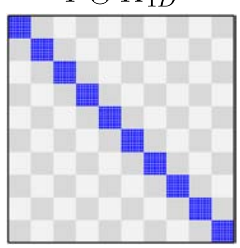
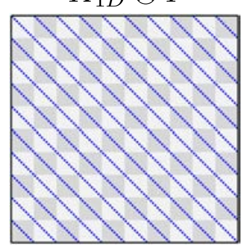
2D discretization:

$K_{2D} =$  $=$ $I \otimes K_{1D}$  $+$ $K_{1D} \otimes I$ 

• Spectral (`cheb.m` (interior))

$K_{1D} =$  full

2D discretization:

$K_{2D} =$  $=$ $I \otimes K_{1D}$  $+$ $K_{1D} \otimes I$ 

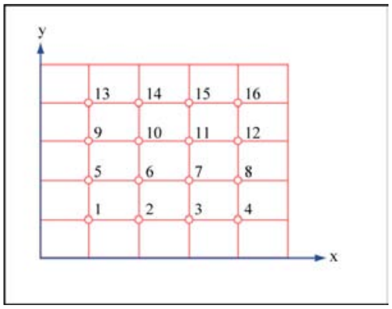
Kronecker product:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left[\begin{array}{cc|cc} a & b & 2a & 2b \\ c & d & 2c & 2d \\ \hline 3a & 3b & 4a & 4b \\ 3c & 3d & 4c & 4d \end{array}\right]$$

Matlab:

```
K2D = kron(I1D, K1D) + kron(K1D, I1D)
```

Lexicographic ordering:



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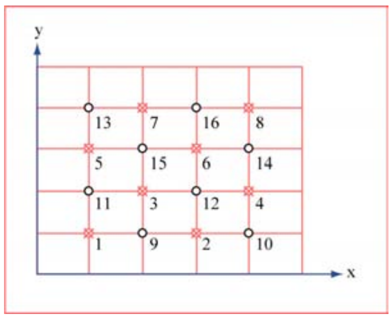
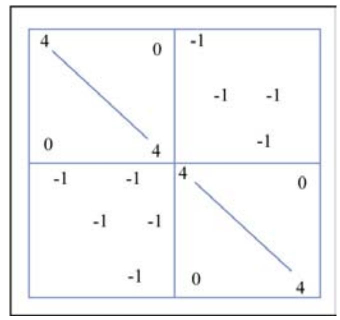
Also possible: red-black ordering



$\rightarrow \tilde{K}_{2D} = \frac{1}{h^2}$



(advantageous for elimination)

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3D:

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K3D = kron(K2D, I1D) + kron(I2D, K1D)
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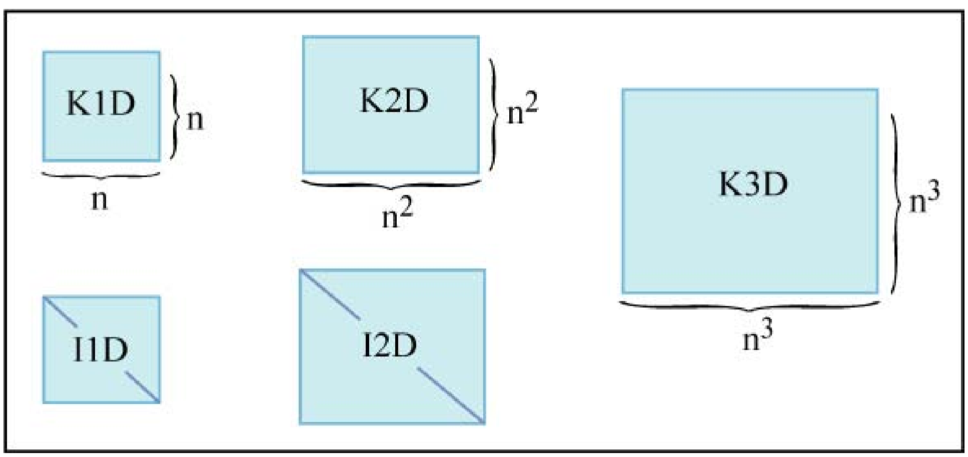

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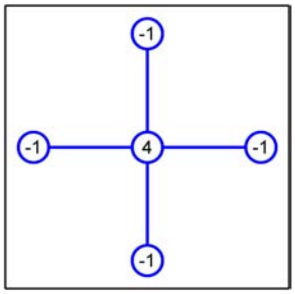
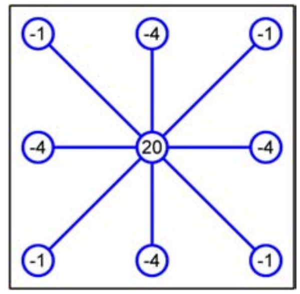
Alternative to Tensor product: directly by Taylor expansion

Ex.: 2D Poisson equation:

$$-(u_{xx} + u_{yy}) = f$$

$\frac{1}{h^2}$ 

5-point stencil

$\frac{1}{6h^2}$ 

9-point stencil

$$\nabla_5{}^2 u = \nabla^2 u + \frac{h^2}{12}(u_{xxxx} + u_{yyyy}) + O(h^4)$$

$$\nabla_9{}^2 u = \nabla^2 u + \frac{h^2}{12}\underbrace{(u_{xxxx} + 2u_{xxyy} + u_{yyyy})}_{=\nabla^2(\nabla^2 u)=\nabla^2 f} + O(h^4)$$

Advantage of 9-point stencil: deferred correction

$$-\nabla_9{}^2 u_{ij} = f(x_i, y_i) - \frac{h^2}{12}\nabla^2 f(x_i, y_i).$$

Fourth order scheme.