18.336 Numerical Methods of Applied Mathematics -- II
Spring 2009

MIT Dept. of Mathematics
Benjamin Seibold

18.336 spring 2009
Problem Set 5

Out Thu 04/16/09 Due Thu 05/07/09

Exercise 9
Solve the following three equations:

- Inviscid Burgers' equation

$$\begin{cases} u_t + u u_x = 0 \text{ on } (x,t) \in]0,1[\times]0,1[\\ u(0,t) = 1 \\ u(x,0) = \cos^2(\frac{3\pi}{2}x)\exp(-2x) \end{cases}$$

  with a non-conservative upwind method

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + U_j^n \frac{U_j^n - U_{j-1}^n}{\Delta x} = 0 \ .$$

- Inviscid Burgers' equation

$$\begin{cases} u_t + \left(\frac{1}{2}u^2\right)_x = 0 \text{ on } (x,t) \in]0,1[\times]0,1[\\ u(0,t) = 1 \\ u(x,0) = \cos^2(\frac{3\pi}{2}x)\exp(-2x) \end{cases}$$

  with a conservative upwind method

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{\frac{1}{2}(U_j^n)^2 - \frac{1}{2}(U_{j-1}^n)^2}{\Delta x} = 0 \ .$$

- Viscous Burgers' equation

$$\begin{cases} u_t + u u_x = 10^{-3} u_{xx} \text{ on } (x,t) \in]0,1[\times]0,1[\\ u(0,t) = 1 \ , \ u(1,t) = 0 \\ u(x,0) = \cos^2(\frac{3\pi}{2}x)\exp(-2x) \end{cases}$$

  with a non-conservative treatment of the nonlinear advection

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + U_j^n \frac{U_j^n - U_{j-1}^n}{\Delta x} = 10^{-3} \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2} \ .$$

  (I strongly recommend an implicit treatment of the diffusion term.)

For the resolutions $\Delta x \in \{\frac{1}{100}, \frac{1}{500}, \frac{1}{2500}\}$, plot the solutions at $t \in \{0.1, 0.5, 1.0\}$. Explain your observations.

Exercise 10
Consider the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \text{ on } x \in]-1,1[\tag{1}$$

with periodic boundary conditions. Unlike viscous Burgers' equation, here the dispersive term $u_{xxx} = 0$ prevents waves from breaking. Although the KdV equation is nonlinear, it possesses smooth traveling wave solutions, so called *solitons*

$$u(x,t) = f_c(x - ct) \ , \tag{2}$$

where

$$f_c(x) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}x\right) \ .$$

Here $\text{sech}(x) = \frac{2}{e^x + e^{-x}}$. It is a nice exercise in differentiation to verify that for every $c > 0$, (2) solves (1).

Write two methods for (1):

- A simple first order accurate finite difference method. I recommend an explicit (conservative) upwind treatment of the nonlinear advection term, and an implicit treatment of the dispersion term.
- A spectral method. Note that soliton solutions are smooth, so you can treat the equation in non-conservative form, and approximate derivatives with spectral accuracy.

Test and compare your two codes on the following test cases, with initial data

(a) $u_0(x) = f_{400}(x)$

(b) $u_0(x) = f_{400}(x + 0.7) + f_{200}(x)$

(c) $u_0(x) = \frac{1}{2}\left(f_{400}(x + 0.7) + f_{200}(x)\right)$

Plot the results at time $t = 0.015$. Explain how the three cases behave. In particular explain how the nonlinear nature of (1) becomes visible in cases (b) and (c). Then run your codes for case (b) up to $t = 0.1$, and see if you can get anything close to the true solution.

Exercise 11
Write a good (preferably high order) level set method for the level set equation

$$\phi_t - \kappa|\nabla\phi| = 0$$

where

$$\kappa = \nabla \cdot \left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$

is the curvature. Use the solution at $t = \frac{T}{2}$ of the problem given in Exercise 8 as initial condition. Show the evolution of the zero contour until it vanishes. In which point does the zero contour vanish?