18.335 Midterm, Fall 2010

You have 2 hours.

Problem 1: SVD Stability (30 points)

Consider the problem of computing the SVD $A = U\Sigma V^*$ from a matrix A (the input).

(a) Explain what it would mean for this computation (outputs U, Σ, and V) to be backwards stable.

(b) Explain why this algorithm cannot be backwards stable. (Hint: think about e.g. what property the computed $\tilde{U}$ would have to have.)

(c) Practical SVD algorithms are, however, stable (for the general definition of stability). Write down what this means.

Problem 2: Least squares (30 points)

Suppose that we want to solve the **weighted least-squares** problem

$$\min_x \|B^{-1}(Ax-b)\|_2$$

where B ($m\times m$) is a nonsingular square matrix and A ($m\times n$) has full column rank.

(a) Write down the equivalent of the normal equations ($A^*Ax = A^*b$ for ordinary least-squares) that the optimum x must satisfy. [*Reminder:* from class, $x^*Cx - x^*d - d^*x$ for $C = C^*$ positive-semidefinite is minimized when $Cx = d$.]

(b) Give a stable way to solve this that avoids squaring the condition number of A. (Just re-express it in terms of stable algorithms considered in class. You don't have to write out the in-class parts of the algorithm, just say "___ factorization of ___" etcetera.)

Problem 3: Eigenvalues (30 points)

(a) Recall the power method: computing $x_{n+1} = Ax_n/\|Ax_n\|$ for $n = 1,2,\ldots$, with some random x_1, which converges to an eigenvector q_1 corresponding to the eigenvalue λ_1 with largest magnitude. (Number the eigenvalues in order $|\lambda_1| \geq |\lambda_2| \geq \cdots$.) Explain why $|\lambda_1| = |\lambda_2|$ is generally a problem for the convergence of this algorithm, but $\lambda_1 = \lambda_2$ is not a problem (assume A is diagonalizable).

(b) Compare and contrast the convergence properties and computational cost of the following two algorithms for computing the eigenvalue closest to μ of a *large sparse* Hermitian matrix A, where μ is in the middle of the spectrum somewhere: Lanczos for the smallest-magnitude eigenvalue of $(A-\mu I)^2$, or Lanczos for the largest-magnitude eigenvalue of $(A-\mu I)^{-1}$.