18.330 :: Homework 4 :: Spring 2012 :: Due Tuesday April 3

1. (1pt) Compute $3^{1/3}$ to 10 digits of accuracy using Newton's method. Explain how you obtained your answer.

2. One method to find the solution of the equation $x = \phi(x)$ for some function ϕ is to use the *fixed point iteration* $x_{k+1} = \phi(x_k)$.

   a) (1pt) Convergence occurs when ϕ is a contractive mapping, i.e., for all $x \neq y$ we have

   $$|\phi(x) - \phi(y)| < |x - y|.$$

   Show that if $|\phi'(x)| < 1$ for all x, then ϕ is a contractive mapping.

   b) (.5pt) Find a function ϕ for which $x = \phi(x)$ has a unique solution, yet the fixed point iteration diverges.

   c) (1pt) Consider a function $f(x)$ with a single root x^* such that $f'(x) \neq 0$ in a neighborhood of x^*. Cast Newton's iteration as a fixed-point iteration $x_{k+1} = \phi(x_k)$. Use part a) to find a criterion on f, f', and f'' in a neighborhood of x^*, which guarantees that the iteration will converge to a fixed point.

3. (2.5pts) Use Newton's method in its multivariable form to find a solution of

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 100, \\ x_1 x_2 x_3 &= 1, \\ x_1 - x_2 - \sin x_3 &= 0. \end{aligned}$$

4. Consider Newton's method for minimizing $F(x)$:

   $$x_{k+1} = x_k - \frac{F'(x_k)}{F''(x_k)}.$$

   In what follows we'll take $F(x) = 1 + \int_0^x \text{atan}(y)\, dy$.

   a) (.5pt) Show that F is strictly convex, i.e. $F''(x) > 0$. (Strictly convex functions always have a unique minimum.)

   b) (.5pt) Find one value of the starting guess x_0 for which Newton's method converges, and one for which it diverges. (Convexity does not ensure convergence).

   c) (1pt) Explain briefly how you would design a foolproof method for finding the minimum of a convex function F, in an interval $[a, b]$ for which $F'(a) < 0$ and $F'(b) > 0$.

5. (2pts) You would like to precisely determine the resistance of an electrical component. The advertised value is $R = 2\Omega$ (Ohms). When connecting the resistance to a battery, you measure the voltage and current with a (cheap) multimeter as $V = 2.9V$ (Volts) and $I = 1.4A$ (Amps) respectively. You figure that Ohm's law $V = RI$ is not exactly satisfied because there are errors both in the measured values of $V, I,$ and in the advertised value of R. Find the "best" fit for $V, I,$ and R by finding the minimum value of the function

   $$F(V, I, R) = (V - RI)^2 + 10(R - 2)^2 + 10(V - 2.9)^2 + 10(I - 1.4)^2$$

   using Newton's method.

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