

$$y' = x^2 + y^2, \quad y(0) = 0, \quad y(1) = ?$$

Taylor2

2	.5	1.0	0.250000000		
4	.25	1.0	.321731191	71731191	
8	.125	1.0	.342583377	20852186	3.44
16	.0625	1.0	.348251999	5668621	3.68
32	.03125	1.0	.349728511	1476513	3.84
64	.015625	1.0	.350104981	376470	3.92
128	.0078125	1.0	.350200001	95020	3.96
256	.00390625	1.0	.350223868	23867	3.98

Taylor3

2	.5	1.0	0.341236633		
4	.25	1.0	.348290369	7053737	
8	.125	1.0	.349919660	1629291	4.33
16	.0625	1.0	.350187885	268225	6.07
32	.03125	1.0	.350226024	38139	7.03
64	.015625	1.0	.350231096	5072	7.52
128	.0078125	1.0	.350231749	654	7.76
256	.00390625	1.0	0.350231832	83	7.88

Taylor4

2	.5	1.0	0.345732520		
4	.25	1.0	.349740267	4007747	
8	.125	1.0	.350191775	451508	8.88
16	.0625	1.0	.350228999	37224	12.13
32	.03125	1.0	.350231655	2656	14.01
64	.015625	1.0	.350231832	177	15.00
128	.0078125	1.0	.350231844	11	15.50
256	.00390625	1.0	0.350231844	1	15.75

Standard RK4, with $p = q = 1/2$:

2	.5	1.0	0.350786879		
4	.25	1.0	.350289294	-497584	
8	.125	1.0	.350236294	-53000	9.39
16	.0625	1.0	.350232151	-4142	12.79
32	.03125	1.0	.350231864	-287	14.44
64	.015625	1.0	.350231846	-19	15.23
128	.0078125	1.0	.350231844	-1	15.62
256	.00390625	1.0	0.350231844	0	15.81

Alternative "1,3,3,1" RK4, with $p = 1/3$, $q = 2/3$:

2	.5	1.0	0.350631794		
4	.25	1.0	.350274818	-356976	
8	.125	1.0	.350235410	-39408	9.06
16	.0625	1.0	.350232101	-3309	11.91
32	.03125	1.0	.350231862	-240	13.81
64	.015625	1.0	.350231845	-16	14.88
128	.0078125	1.0	.350231844	-1	15.43
256	.00390625	1.0	0.350231844	0	15.71

Program RK4A

c ... This short FORTRAN program was all it took to grind out our y(1)
c results via the 1,3,3,1 variant of 4th-order Runge-Kutta

implicit double precision (a-h,o-z)

nsteps = 2
h = 0.5d0

do 59 iter=1,8

h3 = h / 3

yprev = y

x = 0

y = 0

do 49 istep=1,nsteps

sA = x*x + y*y

sB = (x+h3) ** 2 + (y + h3*sA)**2

sC = (x+2*h3)**2 + (y - h3*sA + h*sB)**2

sD = (x+h)*(x+h) + (y + h*(sA - sB + sC))**2

x = x + h

y = y + h * (sA + 3*sB + 3*sC + sD) / 8

49 continue

chprev = change

change = y - yprev

ratio = chprev / change

55 write (*,55) nsteps, h, x,y, change, ratio
format (i4, f12.8, f10.3, 2x, 2f15.9, f8.2)

nsteps = 2*nsteps

h = h / 2

59 continue

end