on AITKEN EXTRAPOLATIONS

Employed **passively** — though repetitively — such extrapolations applied for instance to the sequence

$$\mathbf{x}_{n+1} = \mathbf{cos}(\mathbf{x}_n)$$

begun from the initial estimate $x_0 = 0.8$ behave commendably enough:

1	.800 000 000 000			
2	.696 706 709 347			
3	.766 959 631 892	.738 520 675 863		
4	.720 023 855 606	.738 822 315 846		
5	.751 789 998 906	.738 968 356 620	.739 105 426 131	
6	.730 467 564 690	.739 031 482 244	.739 079 541 548	
7	.744 862 516 208	.739 061 010 164	.739 086 961 286	.739 085 308 272
8	.735 181 105 569	.739 074 123 599	.739 084 599 842	.739 085 169 959
9	.741 709 294 169	.739 080 157 830	.739 085 301 340	.739 085 140 678
10	.737 314 924 109	.739 082 869 590	.739 085 082 894	.739 085 134 765
11	.740 276 408 727	.739 084 107 964	.739 085 148 813	.739 085 133 532
12	.738 282 151 458	.739 084 667 438	.739 085 128 493	.739 085 133 281
13	.739 625 793 054	.739 084 922 042	.739 085 134 668	.739 085 133 229
14	.738 720 830 240	.739 085 037 342	.739 085 132 773	.739 085 133 218
15	.739 330 483 032	.739 085 089 728	.739 085 133 351	.739 085 133 216
16	.738 919 840 384	.739 085 113 478	.739 085 133 174	**.739 085 133 215**
17	.739 196 466 358	.739 085 124 261	.739 085 133 228	.739 085 133 215
18	.739 010 133 291	.739 085 129 152	.739 085 133 211	.739 085 133 215

But the Aitken strategy becomes really impressive when employed **actively** on some such sequence as soon as two new estimates have become available from the functional iteration:

1	.800 000 000 000			
2	.696 706 709 347			
3	.766 959 631 892	.**73**8 520 675 863		
4		.739 465 240 716		
5		.738 829 034 844	**.739 085** 085 836	
6			.739 085 165 130	
7			.739 085 111 717	**.739 085 133 215**
8				.739 085 133 215
9				.739 085 133 215

Overleaf, there is yet more on this topic from an old "Problem 7" and its solutions ... illustrating nicely that the original iteration does not even need to be stable for this active strategy to rescue it, and to zoom down onto the fixed point involved!

7 It is alleged that an "active" **Aitken extrapolation** applied right after every two rounds of a simple functional iteration

$$x_1 = g(x_0) \quad , \qquad x_2 = g(x_1) \quad ,$$

followed two iterations onward from that "new and improved" value and then promptly by a fresh extrapolation — and so forth — not only accelerates most already convergent iterations but sometimes even RESCUES iterations which would otherwise have DIVERGED!

Explore this hybrid process in action:

(a) near each of the 2 fixed points of $x_{n+1} = x_n^2 - 2$, and also

(b) near $x = (C-1)/C$ for $x_{n+1} = Cx_n(1-x_n)$, when C=2.5 and 3.5 .

Indeed the "**active**" Aitken extrapolations work like a charm here, though in unstable cases one obviously needs to start fairly close to a desired fixed point for such inferences to remain plausible:

(a)

	x0,x1,x2	geom r		x0,x1,x2	geom r
1	-1.500000000		1	2.500000000	
	0.250000000			4.250000000	
	-1.937500000	-1.2500		16.062500000	6.7500
2	-0.722222222		2	2.195652174	
	-1.478395062			2.820888469	
	0.185651959	-2.2006		5.957411753	5.0165
3	-0.958480660		3	2.039986802	
	-1.081314824			2.161546150	
	-0.830758251	-2.0398		2.672281760	4.2015
4	-0.998889354		4	2.002017694	
	-1.002220059			2.008074847	
	-0.995554953	-2.0011		2.032364592	4.0101
5	-0.999999178		5	2.000005413	
	-1.000001643			2.000021651	
	-0.999996714	-2.0000		2.000086603	4.0000
6	-1.000000000		6	2.000000000	

(b)

<u>C=2.5</u>:	x0,x1,x2	geom r	<u>C=3.5</u>:	x0,x1,x2	geom r
1	0.700000000		1	0.800000000	
	0.525000000			0.560000000	
	0.623437500	-0.5625		0.862400000	-1.2600
2	0.588000000		2	0.693805310	
	0.605640000			0.743538257	
	0.597100476	-0.4841		0.667411911	-1.5307
3	0.599885992		3	0.713457144	
	0.600056972			0.715526166	
	0.599971506	-0.4999		0.712419651	-1.5014
4	0.599999989		4	0.714284276	
	0.600000005			0.714287871	
	0.599999997	-0.5000		0.714282479	-1.5000
5	0.600000000		5	0.714285714	