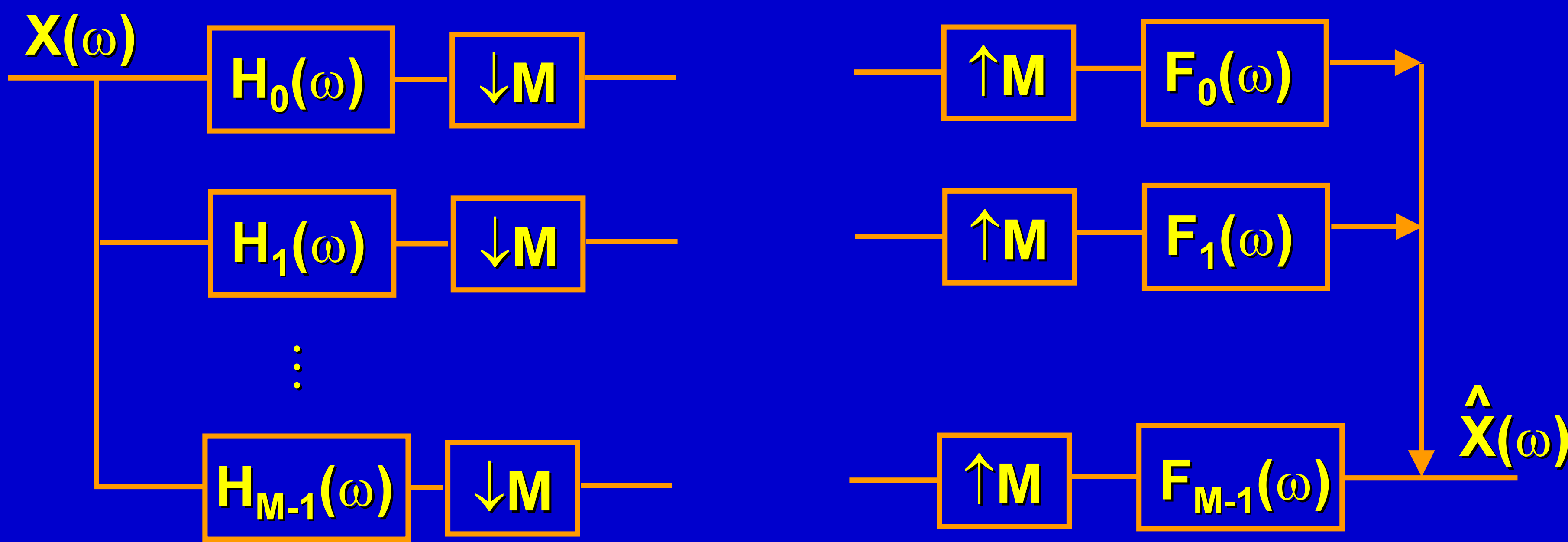
# Course 18.327 and 1.130
# Wavelets and Filter Banks

## M-band wavelets: DFT filter banks and cosine modulated filter banks. Multiwavelets.

## M-channel Filter Banks

- Used in communication e.g. DSL
- 1 Scaling function,   M-1 wavelets




$$(\uparrow 2)(\downarrow 2)X(\omega) = \tfrac{1}{2}\,[X(\omega)+X(\omega+\pi)]$$

$$(\uparrow M)(\downarrow M)X(\omega) = \tfrac{1}{M}\,\Big[\sum_{n=0}^{M-1} X\Big(\omega+\frac{2\pi n}{M}\Big)\Big]$$

## Perfect Reconstruction

$$\sum_{k=0}^{M-1} F_k(\omega) \frac{1}{M} \sum_{n=0}^{M-1} X(\omega + \frac{2\pi n}{M}) H_k(\omega + \frac{2\pi n}{M}) = e^{-i\omega\ell} X(\omega)$$

i.e. $$\frac{1}{M} \sum_{n=0}^{M-1} X(\omega + \frac{2\pi n}{M}) \sum_{k=0}^{M-1} F_k(\omega) H_k(\omega + \frac{2\pi n}{M}) = e^{-i\omega\ell} X(\omega)$$

Matching terms on either side

$n = 0$ $$\sum_{k=0}^{M-1} F_k(\omega) H_k(\omega) = M e^{-i\omega\ell}$$ no distortion

$n \neq 0$ $$\sum_{k=0}^{M-1} F_k(\omega) H_k(\omega + \frac{2\pi n}{M}) = 0$$ no aliasing e.g. $M = 3$

$$F_0(\omega)H_0(\omega) + F_1(\omega)H_1(\omega) + F_2(\omega)H_2(\omega) = 3e^{-i\omega\ell}$$

$$F_0(\omega)H_0(\omega + \tfrac{2\pi}{3}) + F_1(\omega)H_1(\omega + \tfrac{2\pi}{3}) + F_2(\omega)H_2(\omega + \tfrac{2\pi}{3}) = 0$$

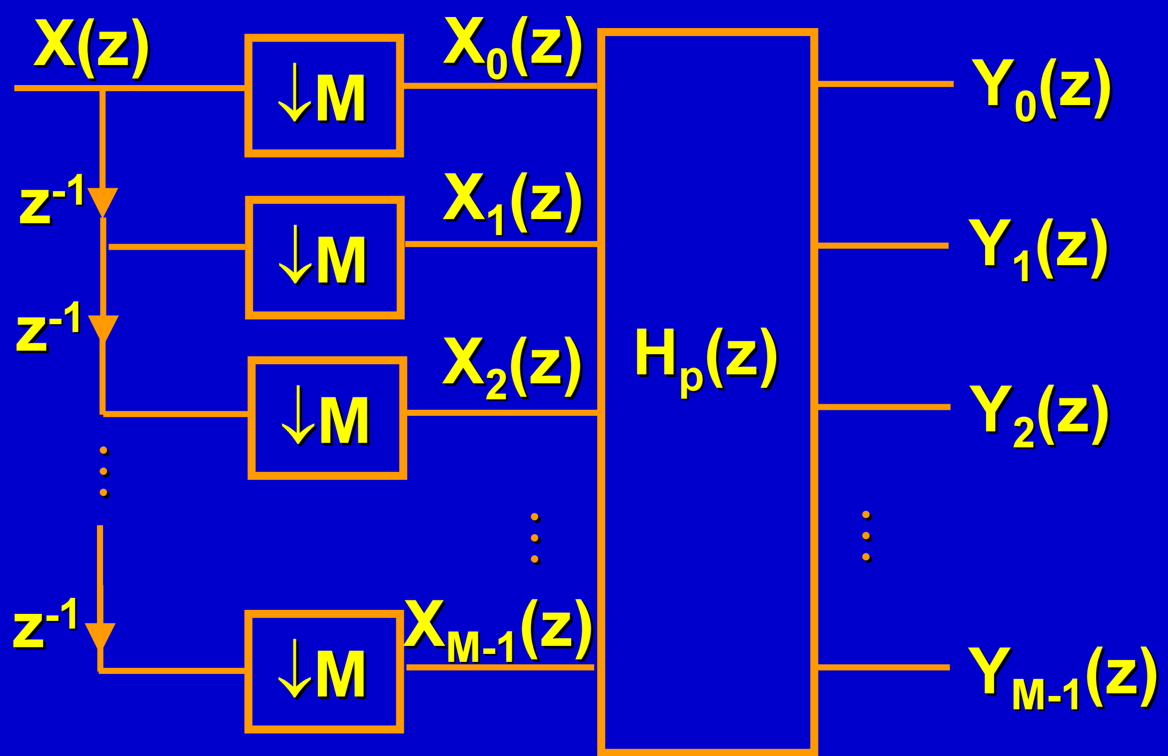
$$F_0(\omega)H_0(\omega + \tfrac{4\pi}{3}) + F_1(\omega)H_1(\omega + \tfrac{4\pi}{3}) + F_2(\omega)H_2(\omega + \tfrac{4\pi}{3}) = 0$$

Cast in matrix form

$$[F_0(\omega) \quad F_1(\omega) \quad F_2(\omega)] \; H_m(\omega) = [3e^{-i\omega\ell} \quad 0 \quad 0]$$

$$H_m(\omega) = \begin{bmatrix} H_0(\omega) & H_0(\omega+\frac{2\pi}{3}) & H_0(\omega+\frac{4\pi}{3}) \\ H_1(\omega) & H_1(\omega+\frac{2\pi}{3}) & H_1(\omega+\frac{4\pi}{3}) \\ H_2(\omega) & H_2(\omega+\frac{2\pi}{3}) & H_2(\omega+\frac{4\pi}{3}) \end{bmatrix}$$

## Polyphase Representation




$$x[Mn] \leftrightarrow X_0(z) = x[0] + z^{-1}x[M] + z^{-2}x[2M] + z^{-3}x[3M] + \ldots$$

$$x[Mn-1] \leftrightarrow X_1(z) = x[-1] + z^{-1}x[M-1] + z^{-2}x[2M-1] + \ldots$$

$$x[Mn-2] \leftrightarrow X_2(z) = x[-2] + z^{-1}x[M-2] + z^{-2}x[2M-2] + \ldots$$

$$\vdots$$

To recover $X(z)$ from $X_0(z)$, $X_1(z)$, $X_2(z)$, …

$$X(z) = \sum_{k=0}^{M-1} z^k X_k(z^M)$$

Much more freedom than 2 channel case

e.g. can have orthogonality & symmetry

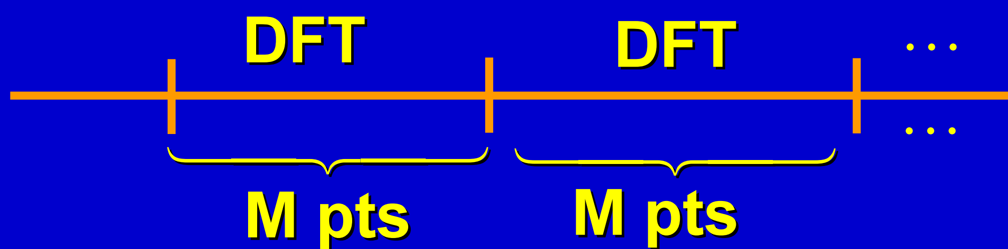
Consider Haar FB ($M = 2$)

Then $H_p(z) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = F_2$ (2 pt DFT matrix)

## M-pt DFT matrix

$$F_M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^{M-1} \\ 1 & w^2 & w^4 & w^{2(M-1)} \\ \vdots & & & \vdots \\ 1 & w^{M-1} & w^{2(M-1)} & w^{(M-1)(M-1)} \end{bmatrix} \qquad w = e^{-i\frac{2\pi}{M}}$$

Suppose $H_p(z) = F_M$




$$Y_0(z) = \sum_{n=0}^{M-1} X_n(z) = \left(\sum_{n=0}^{M-1} x[-n]\right) + \left(\sum_{n=0}^{M-1} x[M-n]\right) z^{-1} + \ldots$$

$$Y_1(z) = \sum_{n=0}^{M-1} w^n X_n(z) = \left(\sum_{n=0}^{M-1} w^n x[-n]\right) + \left(\sum_{n=0}^{M-1} w^{n-M} x[M-n]\right) z^{-1} + \ldots$$

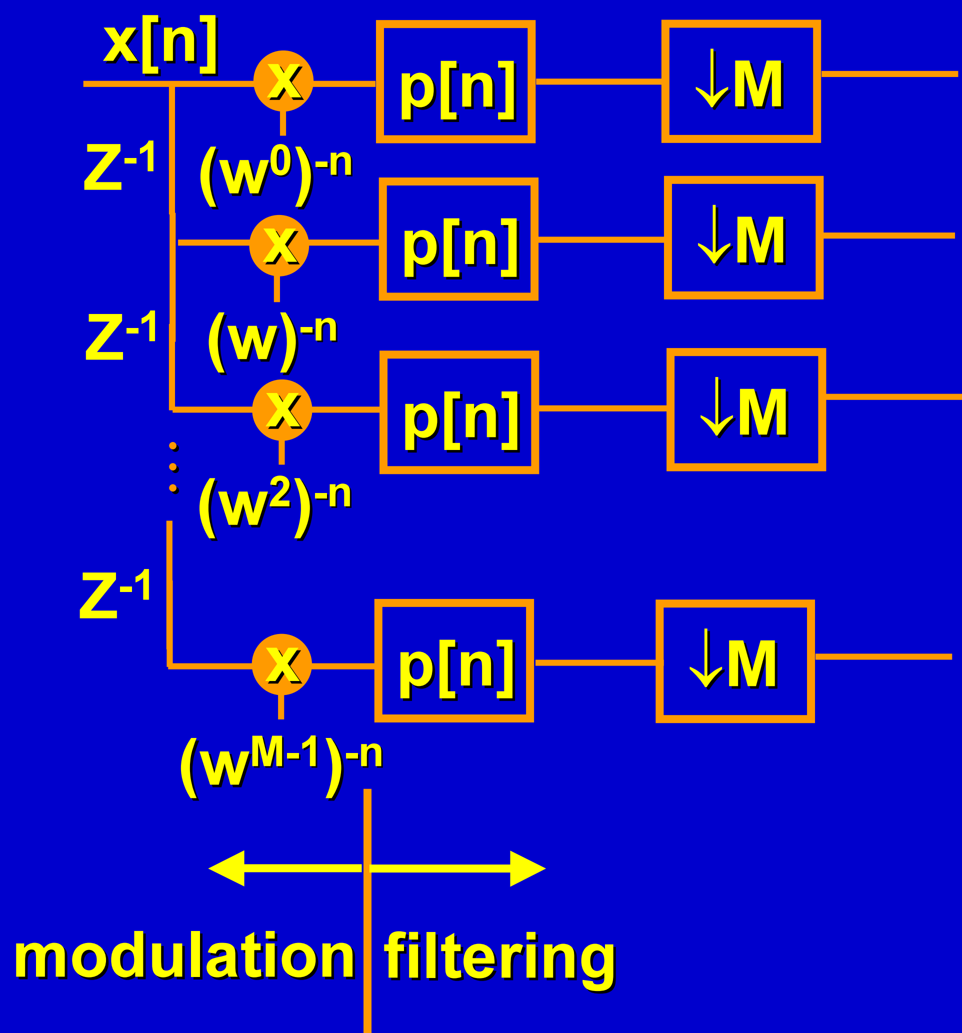
$$\vdots$$

$$Y_k(z) = \sum_{n=0}^{M-1} w^{kn} X_n(z) = \left(\sum_{n=0}^{M-1} w^{kn} x[-n]\right) + \left(\sum_{n=0}^{M-1} w^{k(n-M)} x[M-n]\right) z^{-1} + \ldots$$

Terms in $z^{-k}$ are DFT coefficients of $k^{th}$ block of data.

So filter bank performs a block DFT.

# Modulation followed by filtering




For block DFT,

$$p[n] = [\underset{0}{1}, 1, 1, \ldots, \underset{M-1}{1}]$$

- Can generalize by using other prototype filters.
- p[n] is called the prototype filter.

If $w^{-kn}$ is replaced by $c_{k,n}$ from DCT $\Rightarrow$ Cosine-modulated Filter Bank

$$c_{k,n} = \sqrt{\tfrac{2}{M}}\,\cos\left[\left(k + \tfrac{1}{2}\right)\left(n + \tfrac{M}{2} + \tfrac{1}{2}\right)\tfrac{\pi}{M}\right]$$

## Cosine Modulated Filter Bank (from type IV DCT)

$$h_k[n] = p[n]\sqrt{\tfrac{2}{M}}\cos[(k + ½)(n + \tfrac{M}{2} + \tfrac{1}{2})\tfrac{\pi}{M}]$$

center it!

p[n] chosen to be symmetric LPF.

Only p[n] needs to be designed.

Let L be the length of p[n].

Symmetry: $P[L - 1 - n] = p[n]$

L=2M orthogonality: $p[n]^2 + p[n + M]^2 = 1$

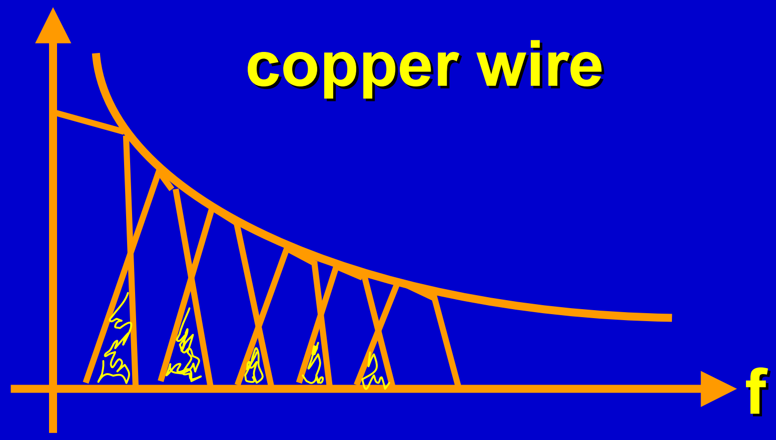
L=4M orthogonality: $p[n]^2+p[n+M]^2+p[n+2M]^2+p[n+3M]^2 = 1$

$p[n]p[n + 2M] + p[n+M]p[n + 3M] = 0$

"Double-shift orthogonality" in M=2 case

Genus of the prototype filter.

## Application to DSL




- assign more bits to lower frequency bands
- orthogonal CMFB can undo the overlaps between channels

### Multiwavelets

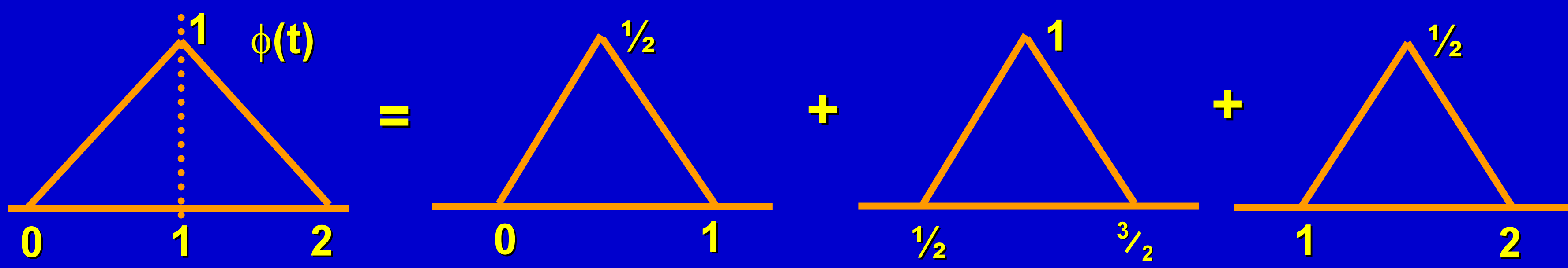
Idea: extend the scalar refinement equation

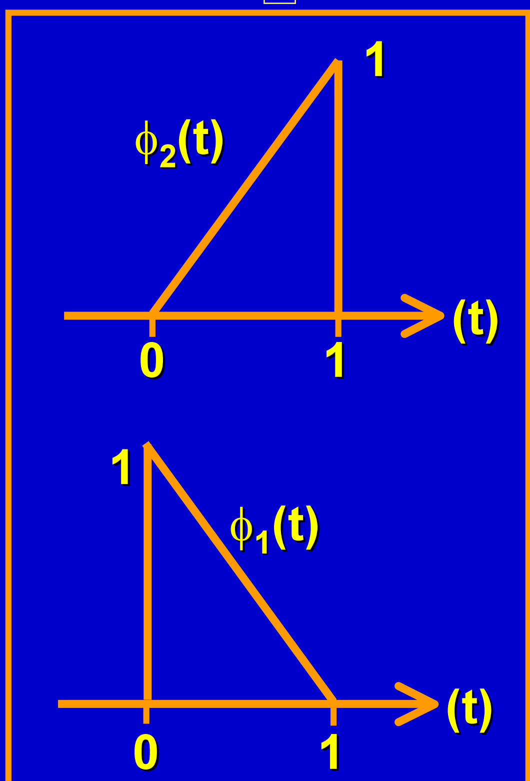
$$\phi(t) = 2 \sum_k h_0[k]\phi(2t - k)$$

into a vector refinement equation

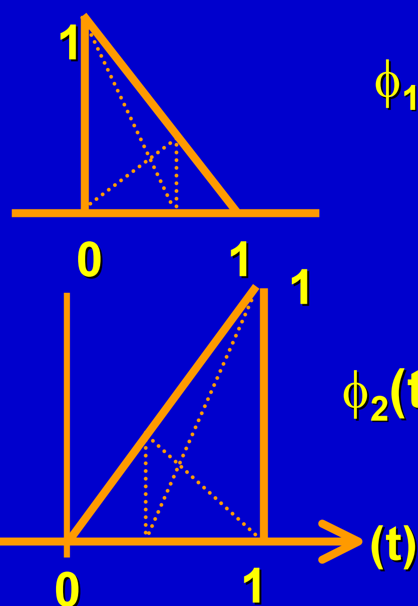
$$\begin{bmatrix} \phi_1(t) \\ \\ \phi_2(t) \end{bmatrix} = 2 \sum_{k=0}^{N-1} \underbrace{\begin{bmatrix} H_0[k] \end{bmatrix}}_{2\text{x}2} \begin{bmatrix} \phi_1(2t - k) \\ \\ \phi_2(2t - k) \end{bmatrix}$$

## e.g. Finite Elements




can use to represent piecewise linear function but allows for representing discontinuous function

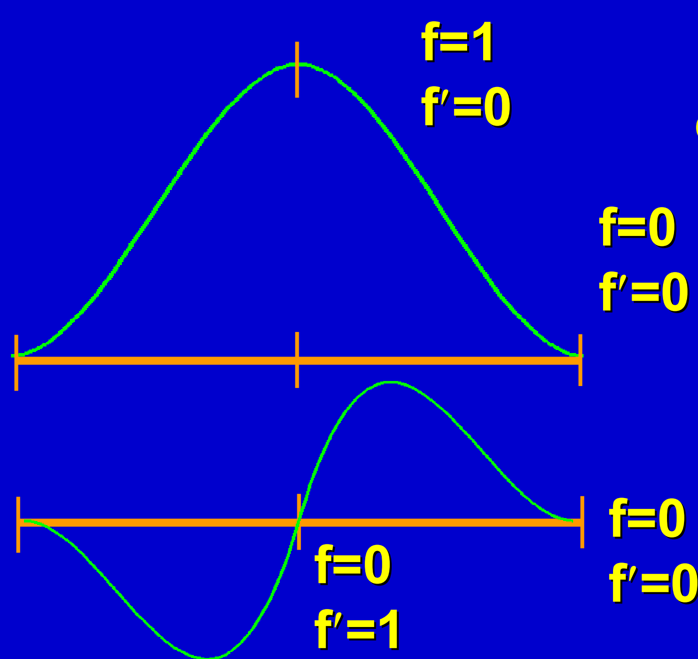



$$\phi_1(t) = \phi_1(2t) + \tfrac{1}{2}\phi_1(2t - 1) + \tfrac{1}{2}\phi_2(2t)$$

$$\phi_2(t) = \phi_2(2t - 1) + \tfrac{1}{2}\phi_2(2t) + \tfrac{1}{2}\phi_1(2t-1)$$

$$\Rightarrow \quad \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} 1 & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_1(2t) \\ \phi_2(2t) \end{bmatrix} \begin{bmatrix} \tfrac{1}{2} & 0 \\ \tfrac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \phi_1(2t-1) \\ \phi_2(2t-1) \end{bmatrix}$$

**Finite Element Multiwavelets**




$\phi_1(t)$

$\phi_2(t)$

$\left.\right\}$ satisfy vector refinement equation

can also come up with orthogonal multiwavelets.