Course 18.327 and 1.130
Wavelets and Filter Banks

Sampling rate change operations: upsampling and downsampling; fractional sampling; interpolation

Downsampling

Definition:

$$(\downarrow 2)\begin{bmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x[0] \\ x[2] \\ x[4] \\ \vdots \\ \vdots \end{bmatrix}$$

As a matrix operation:

$$\begin{bmatrix} & & & \vdots & & & \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & & & \vdots & & & \end{bmatrix} \begin{bmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x[0] \\ x[2] \\ x[4] \\ \vdots \\ \vdots \end{bmatrix}$$

Upsampling

Definition:

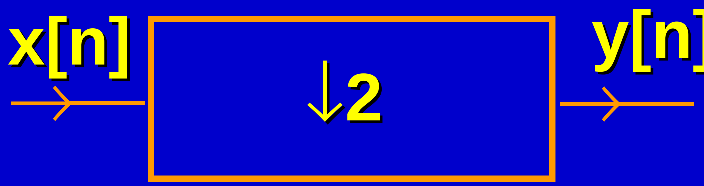
$$(\uparrow 2)\begin{bmatrix} \vdots \\ x[0] \\ x[1] \\ x[2] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x[0] \\ 0 \\ x[1] \\ 0 \\ x[2] \\ 0 \\ \vdots \end{bmatrix}$$

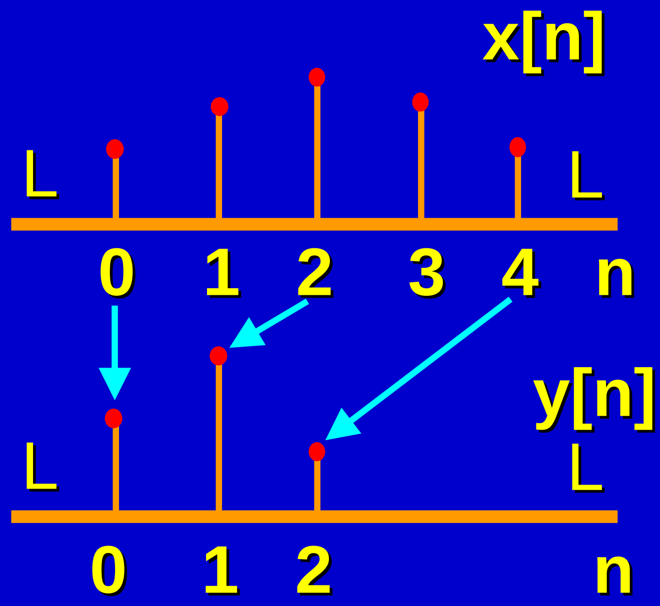
As a matrix operation:

$$
\begin{bmatrix}
 & & \vdots & & \\
\cdots & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & \cdots \\
\cdots & 0 & 0 & 0 & \cdots \\
 & & \vdots & &
\end{bmatrix}
\begin{bmatrix}
\vdots \\ x[0] \\ x[1] \\ x[2] \\ : \\ : \\ :
\end{bmatrix}
=
\begin{bmatrix}
\vdots \\ x[0] \\ 0 \\ x[1] \\ 0 \\ x[2] \\ 0 \\ \vdots
\end{bmatrix}
$$

Downsampling

Downsampling by 2





$$y[n] = x[2n]$$

$$Y(\omega) = \sum_n x[2n]e^{-i\omega n}$$

$$= \sum_{m\ \text{even}} x[m]e^{-i\omega m/2}$$

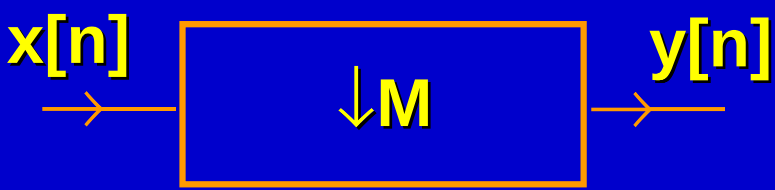
$$= \tfrac{1}{2} \sum_m \{1 + (-1)^m\}\, x[m]e^{-i\omega m/2}$$

$$= \tfrac{1}{2} \{\sum_m x[m]e^{-i\omega \frac{m}{2}} + \sum_m x[m]e^{-i(\frac{\omega}{2} + \pi)m}\} ;$$

$$(-1)^m = e^{-i\pi m}$$

$$= \tfrac{1}{2} \{X(\omega/2) + X(\omega/2 + \pi)\}$$

Downsampling by M



$$y[n] = x[Mn]$$

$$Y(\omega) = \sum_{m = nM} x[m]e^{-i\omega m/M}$$

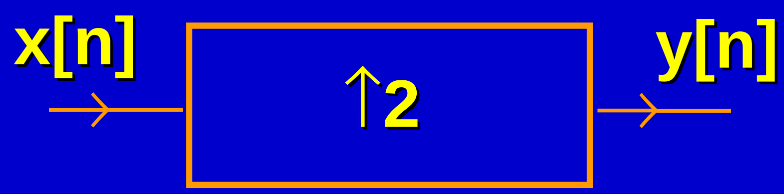
$$= \frac{1}{M} \sum_{m} \{ \sum_{k=0}^{M-1} e^{-i\frac{2\pi}{M}km} \} x[m]e^{-i\omega m/M} \;;$$

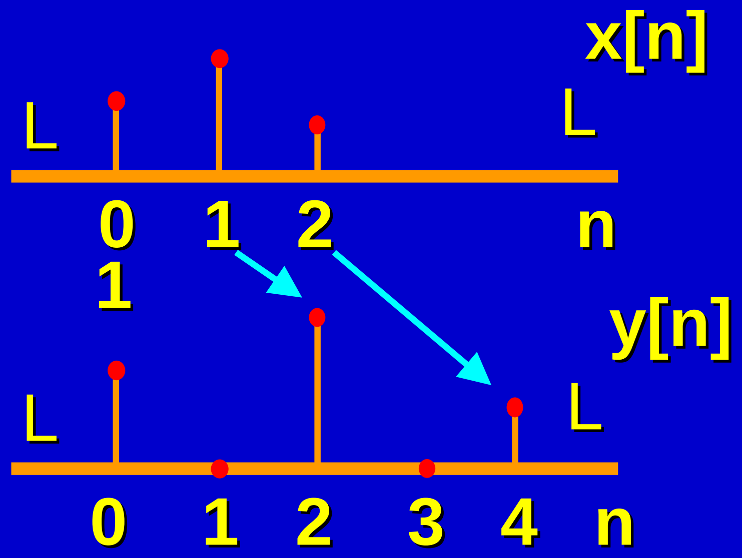
$$\frac{1}{M} \sum_{k=0}^{M-1} (e^{-i\frac{2\pi}{M}m})^k = \begin{cases} 1 \text{ if } m = nM \\ 0 \text{ if } m \neq nM \end{cases}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X(\frac{\omega + 2\pi k}{M})$$

Upsampling

Upsampling by 2





$$y[n] = \begin{cases} x[n/2] & ; n \text{ even} \\ 0 & ; n \text{ odd} \end{cases}$$

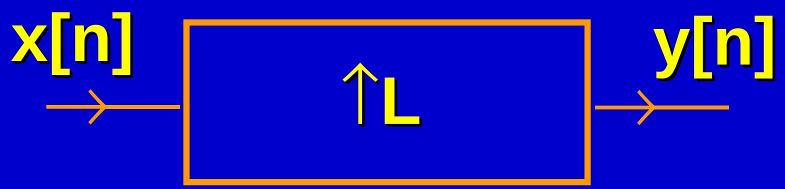
$$Y(\omega) = \sum_{n \text{ even}} x[n/2]e^{-i\omega n}$$

$$= \sum_{m} x[m]e^{-i\omega 2m}$$

$$= X(2\omega)$$

Upsampling by L

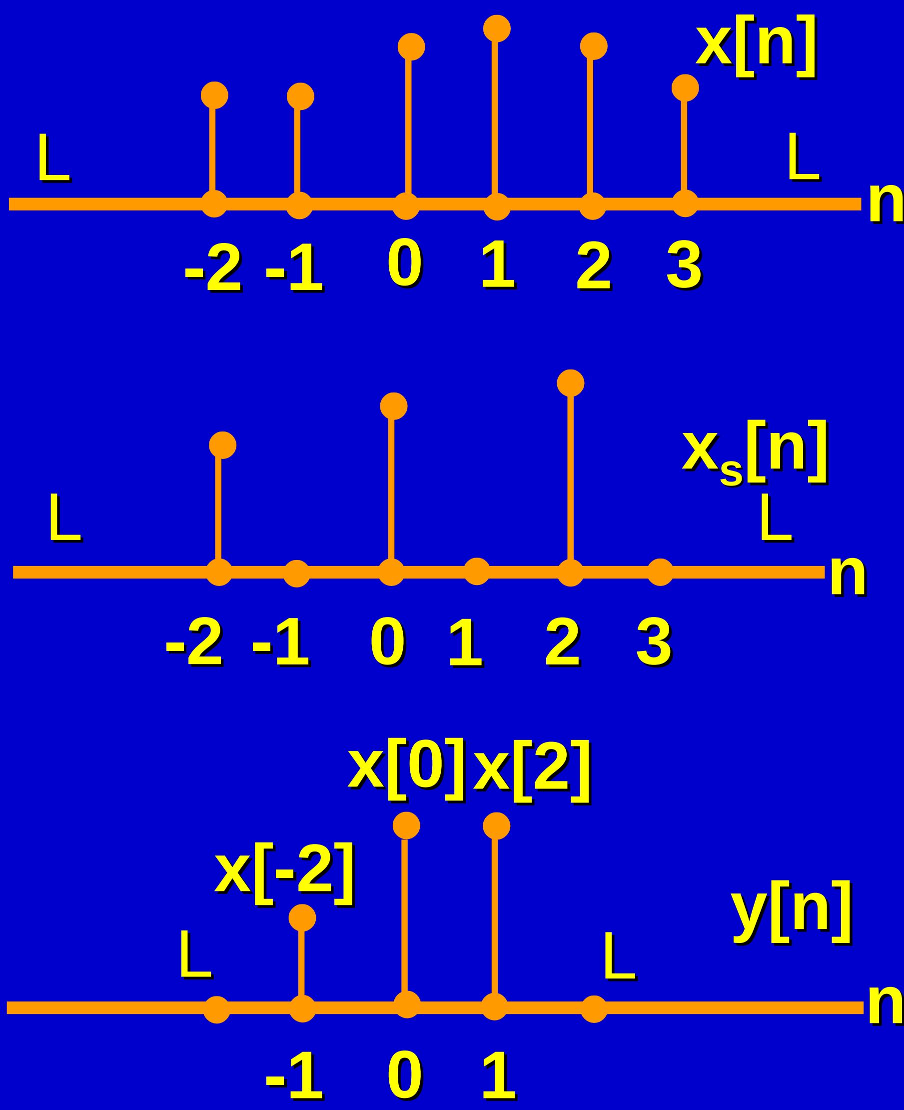
$$y[n] = \begin{cases} x[n/L] & ; \; n = mL \\ 0 & ; \; n \neq mL \end{cases}$$

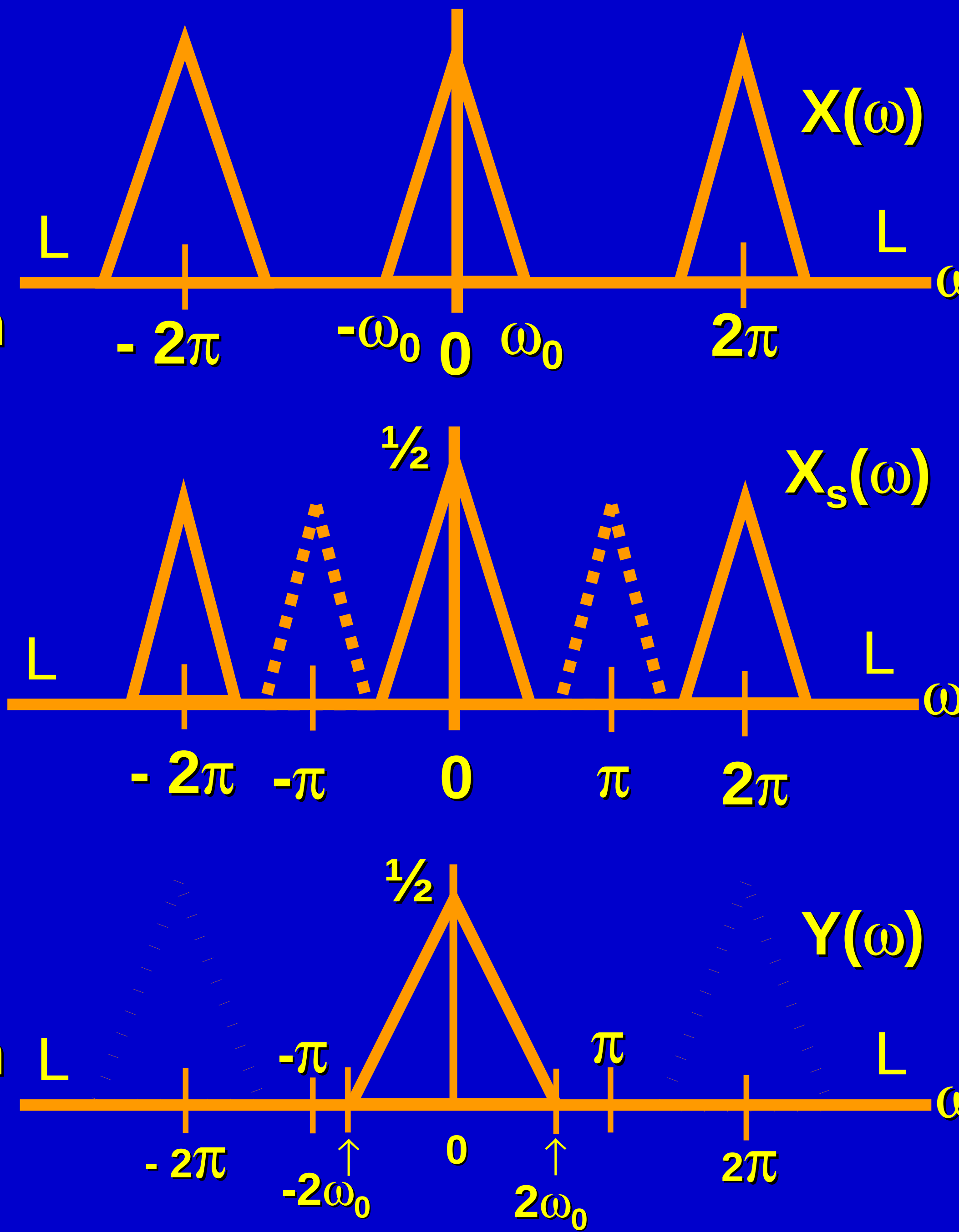


$$Y(\omega) = \sum_{n=mL} x[n/L] \, e^{-i\omega n}$$

$$= X(L\omega)$$

Downsampling

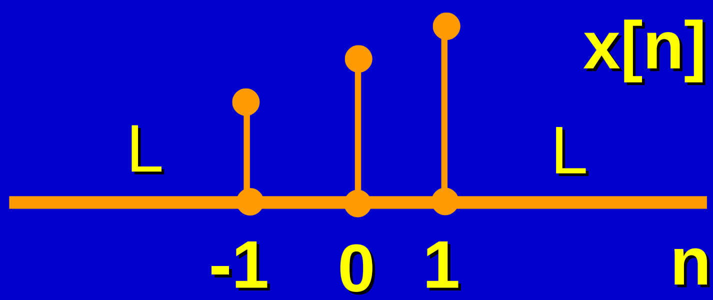


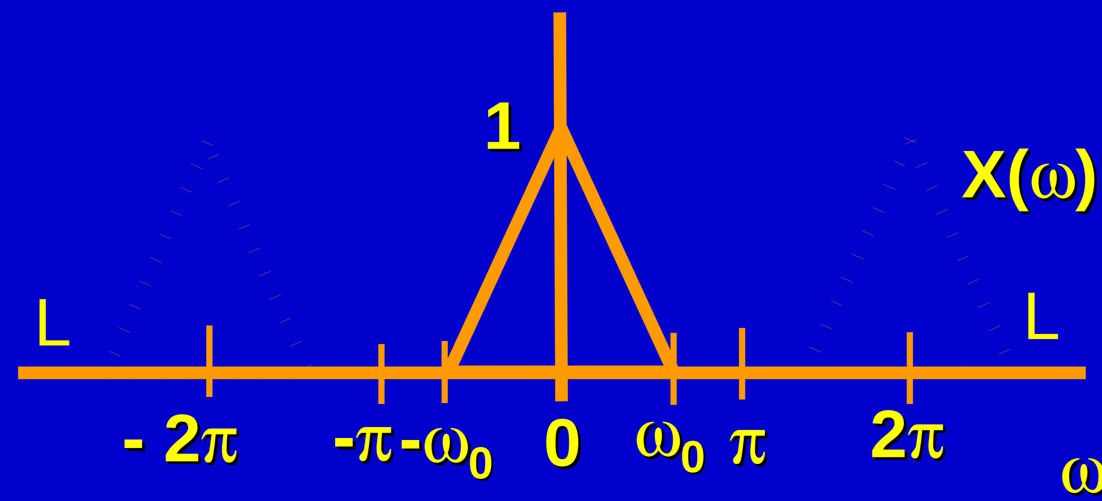


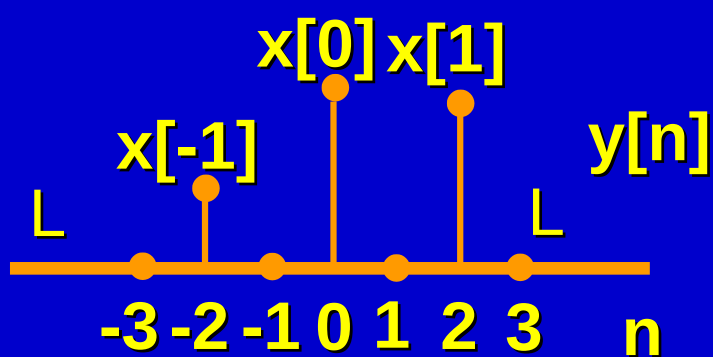
$$y[n] = (\downarrow 2)\, x[n] = x[2n]$$

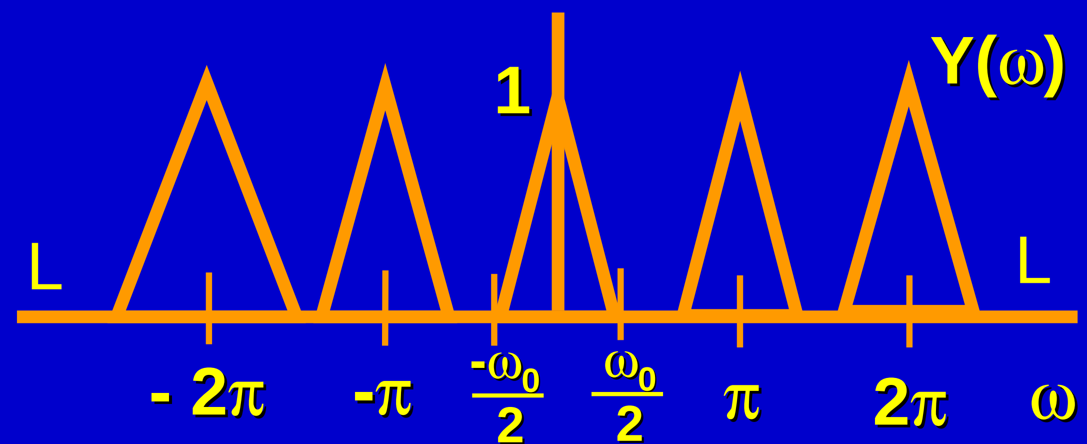
$$Y(\omega) = \tfrac{1}{2}\left\{ X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2} + \pi\right)\right\}$$

Upsampling







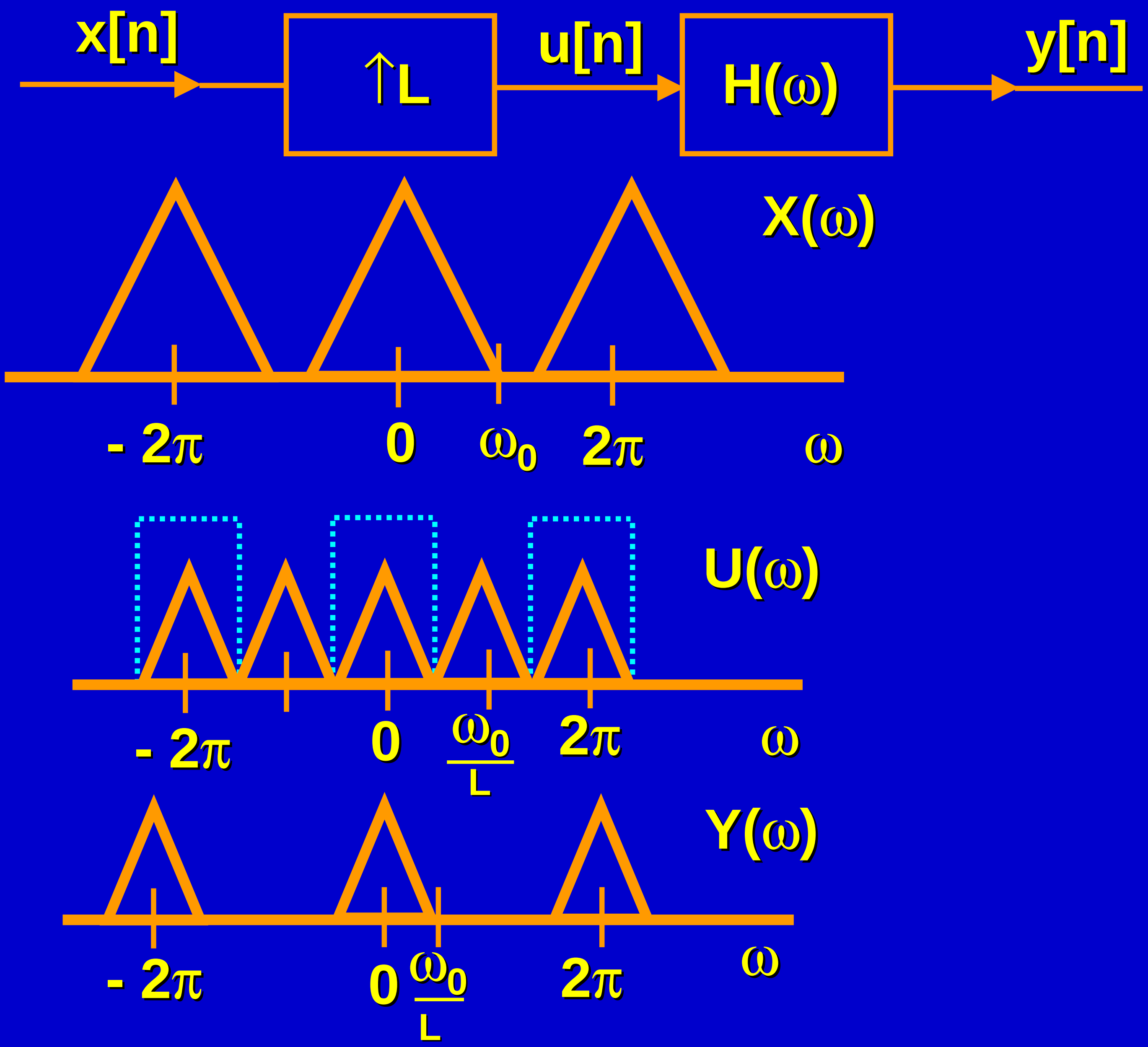


$$y[n] = \begin{cases} x[n/2] & ; n \text{ even} \\ 0 & ; n \text{ odd} \end{cases}$$

$$Y(\omega) = X(2\omega)$$

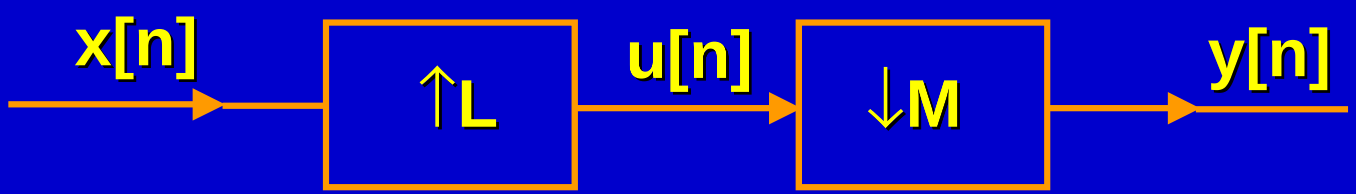
Interpolation

Use lowpass filter after upsampling



Fractional Sampling

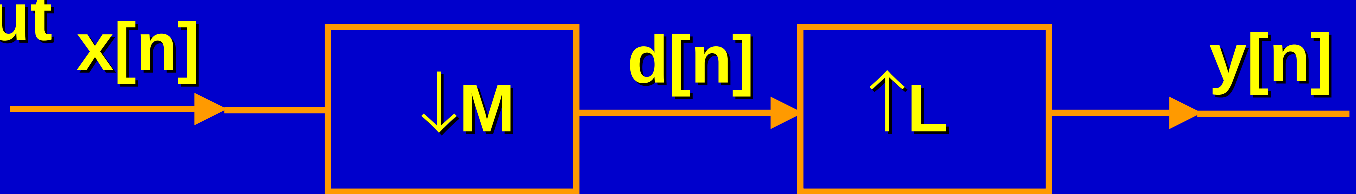
Consider



$$Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} U\left(\frac{\omega + 2\pi k}{M}\right)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega + 2\pi k}{M} L\right)$$

What about



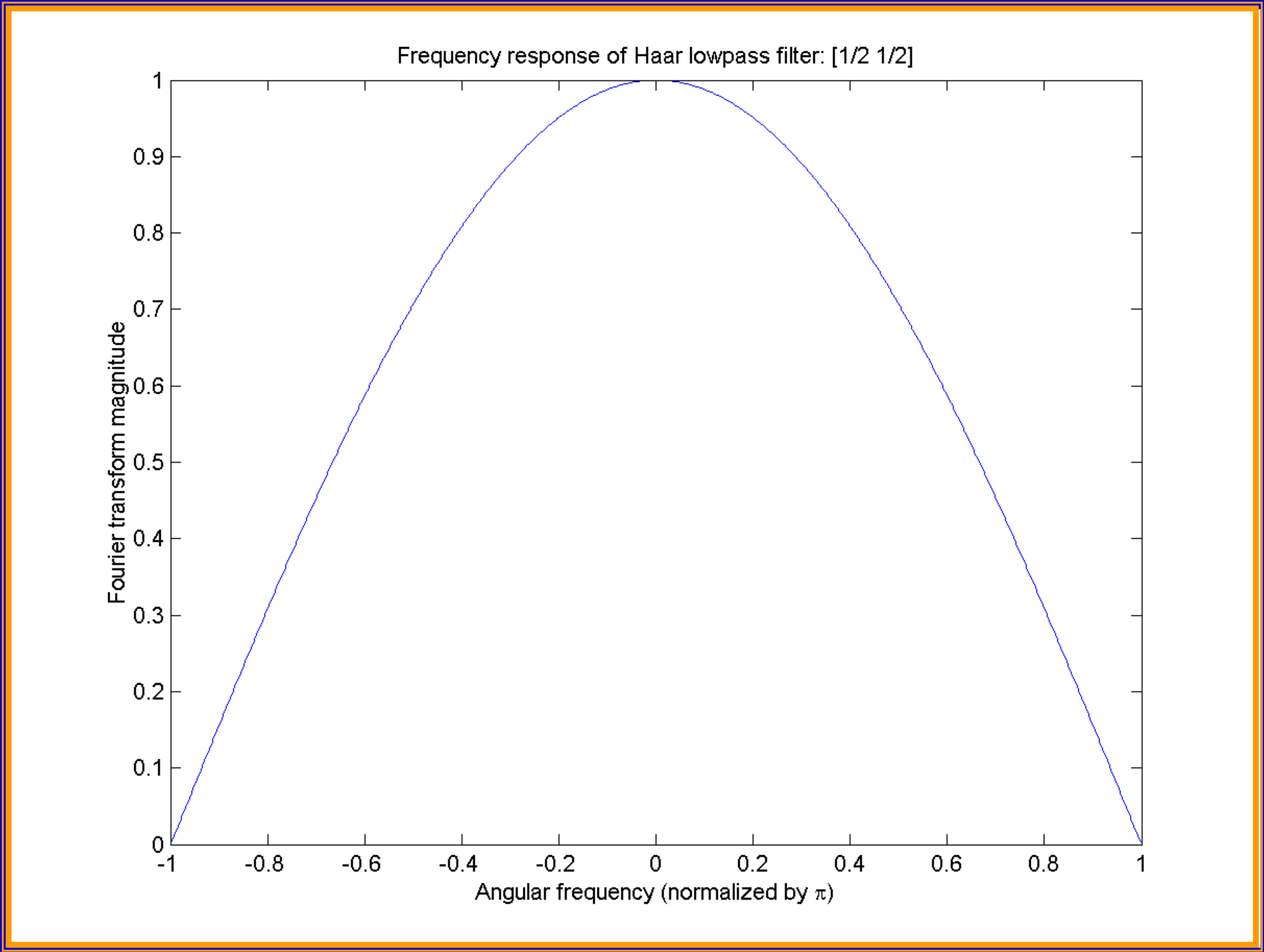
$$Y(\omega) = D(\omega L)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega L + 2\pi k}{M}\right)$$

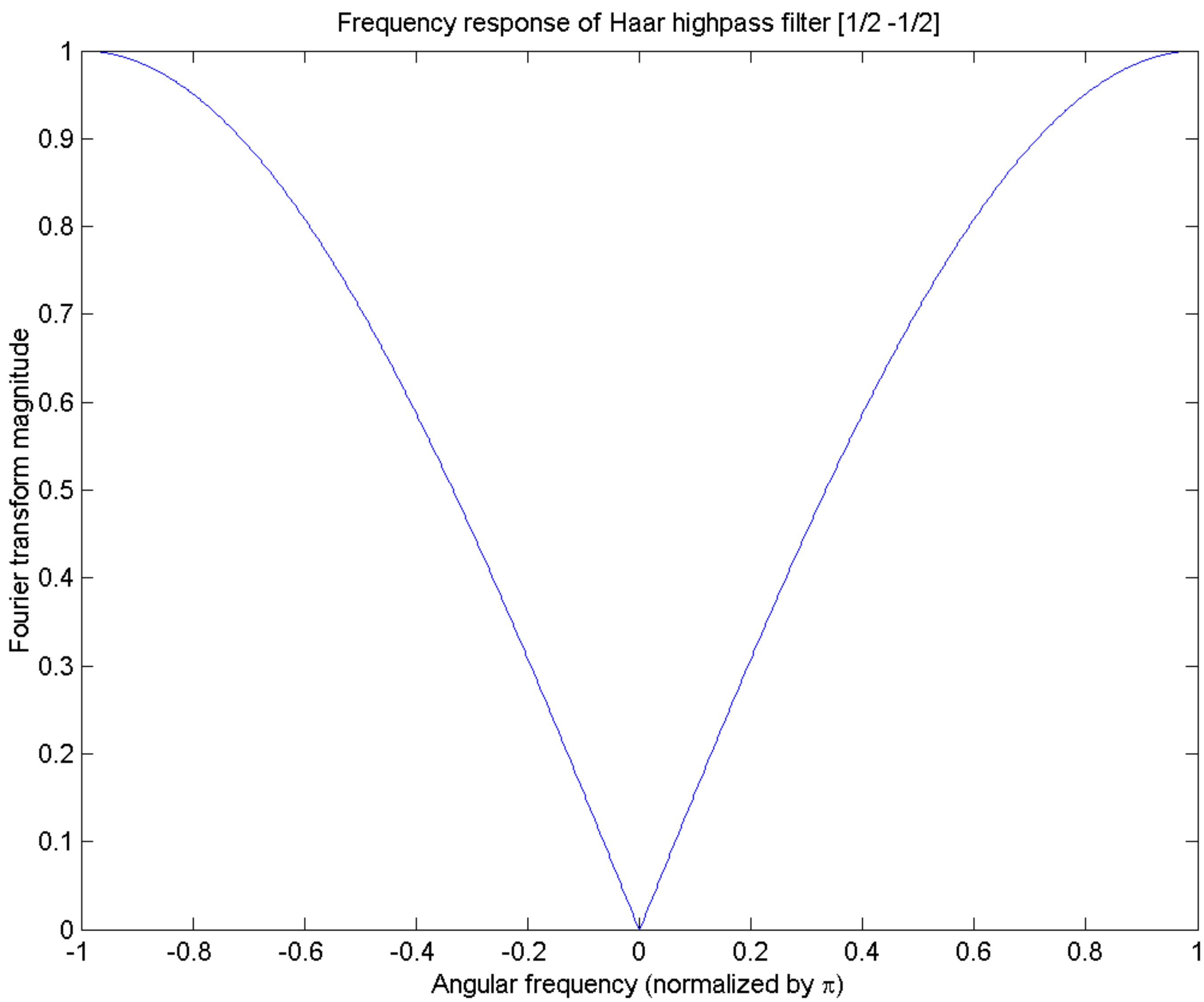
Matlab Example 1

Basic filters, upsampling and downsampling.

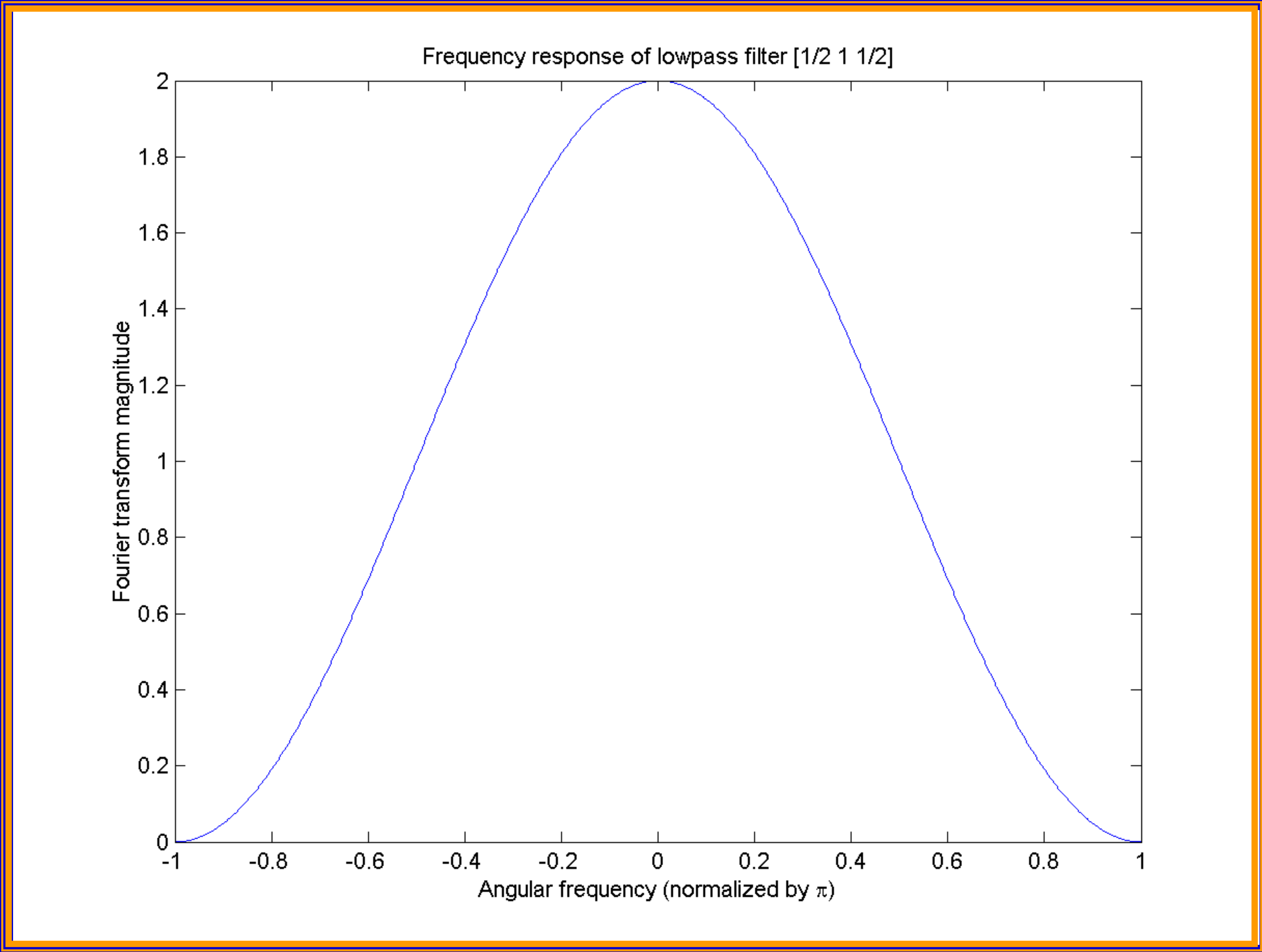
Lowpass filter



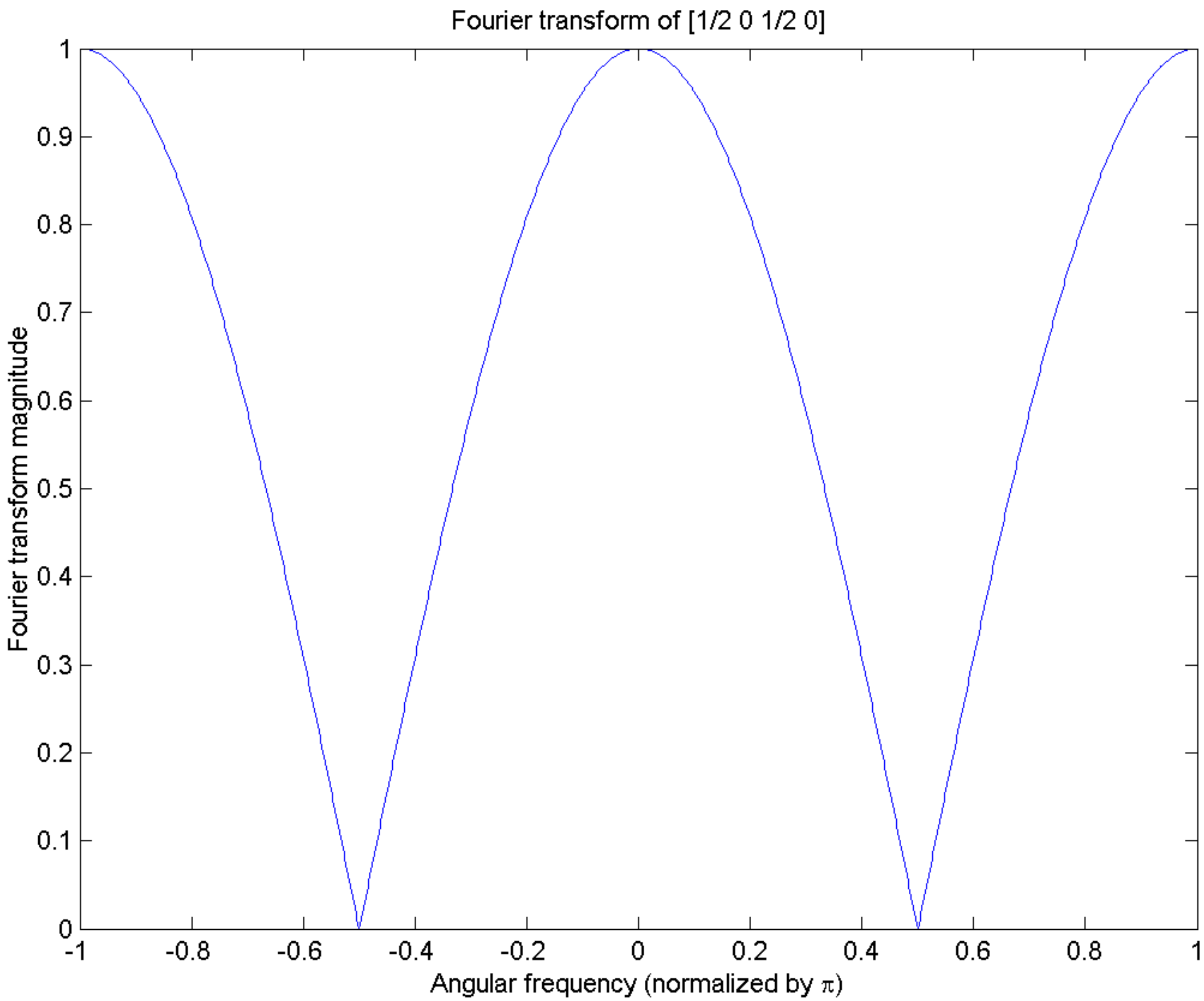
Highpass filter



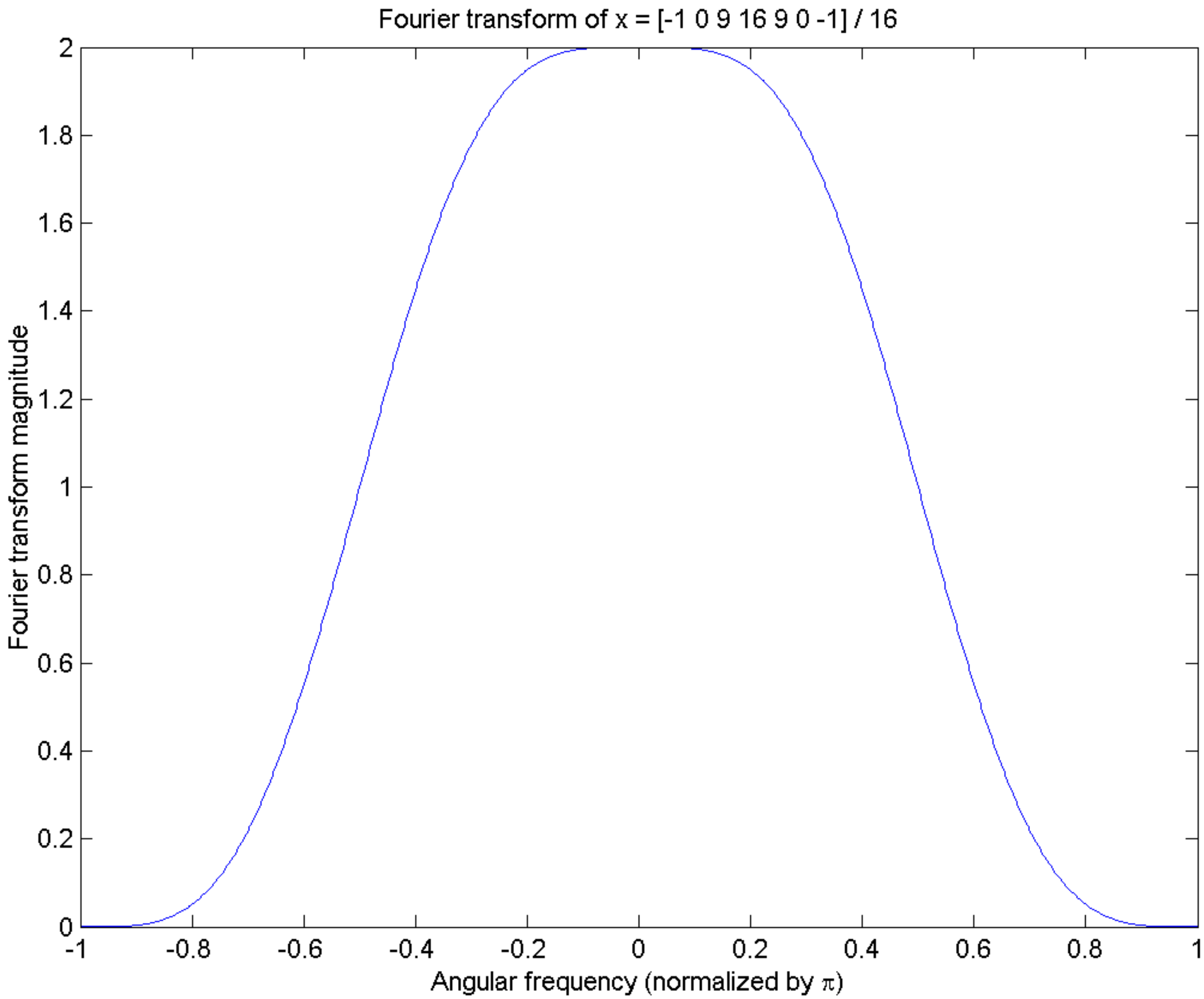
Linear interpolating lowpass filter



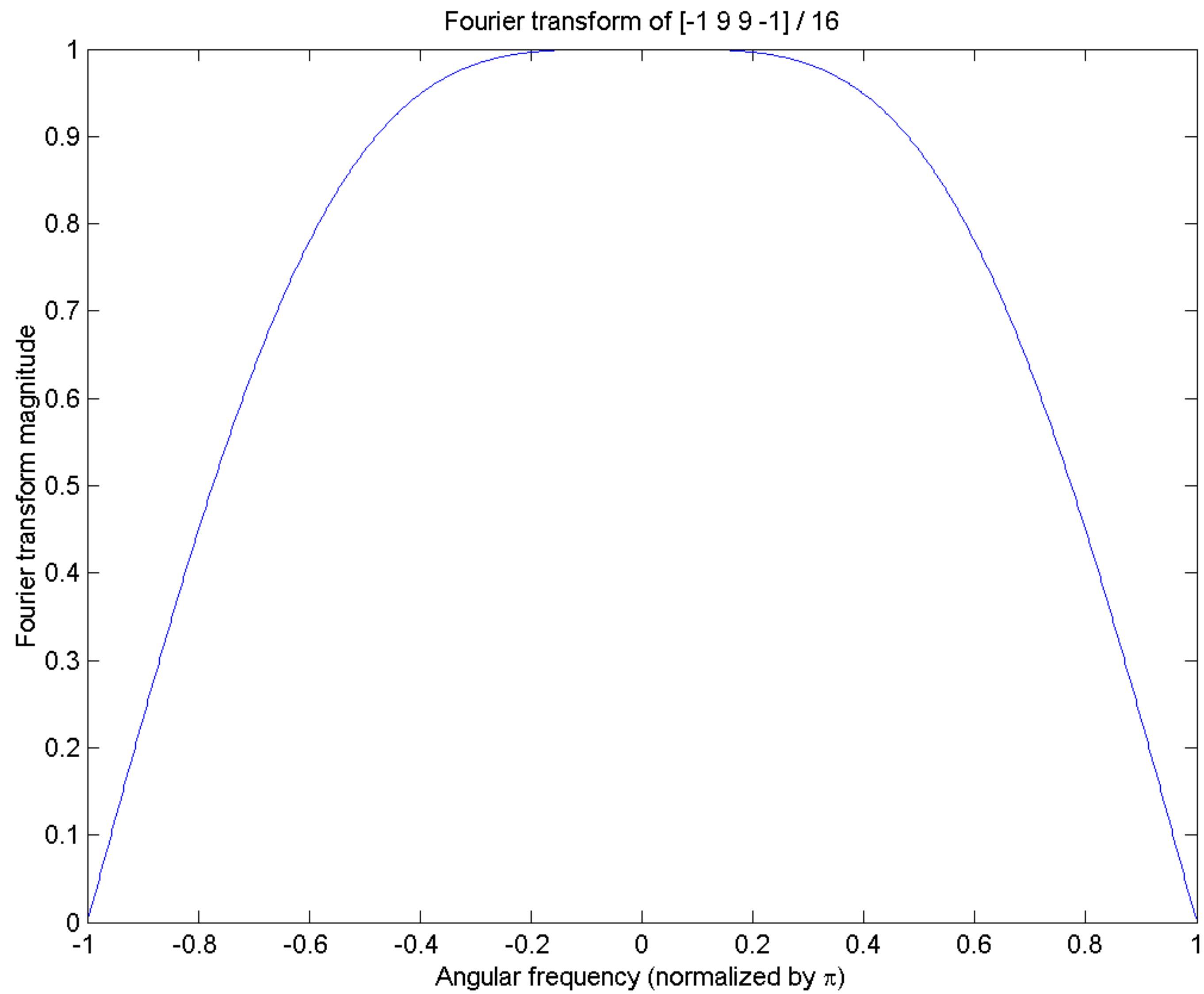
Upsampling


Fourier transform of [1/2 0 1/2 0]
Fourier transform magnitude
Angular frequency (normalized by π)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Downsampling



Downsampling



Downsampling