Course 18.327 and 1.130
Wavelets and Filter Banks

Mallat pyramid algorithm

Pyramid Algorithm for Computing Wavelet Coefficients

Goal: Given the series expansion for a function $f_j(t)$ in V_j

$$f_j(t) = \sum_k a_j[k]\phi_{j,k}(t)$$

how do we find the series

$$f_{j-1}(t) = \sum_k a_{j-1}[k]\phi_{j-1,k}(t)$$

in V_{j-1} and the series

$$g_{j-1}(t) = \sum_k b_{j-1}[k]w_{j-1,k}(t)$$

in W_{j-1} such that

$$f_j(t) = f_{j-1}(t) + g_{j-1}(t) \quad ?$$

Example: suppose that $\phi(t)$ = box on [0,1]. Then functions in V_1 can be written either as a combination of

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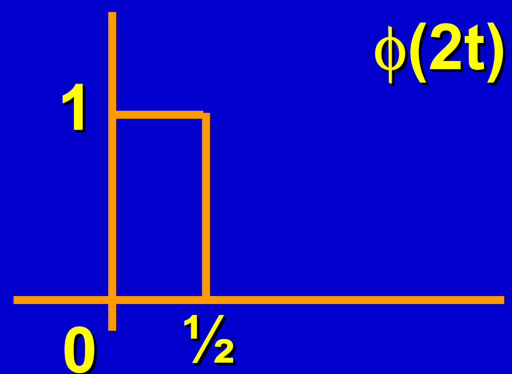

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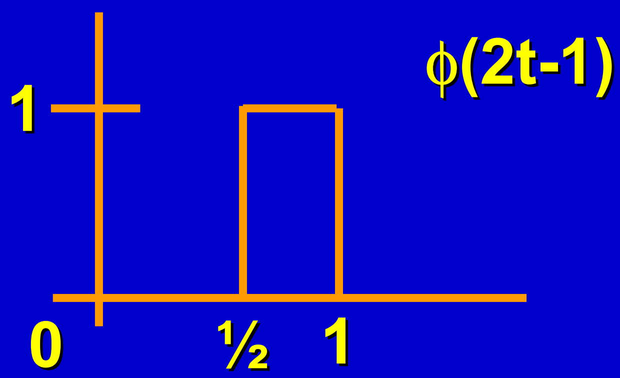

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or as a combination of

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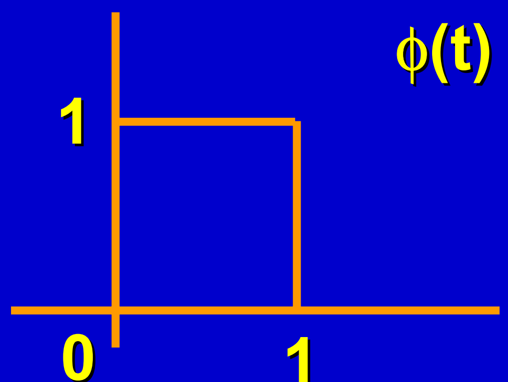

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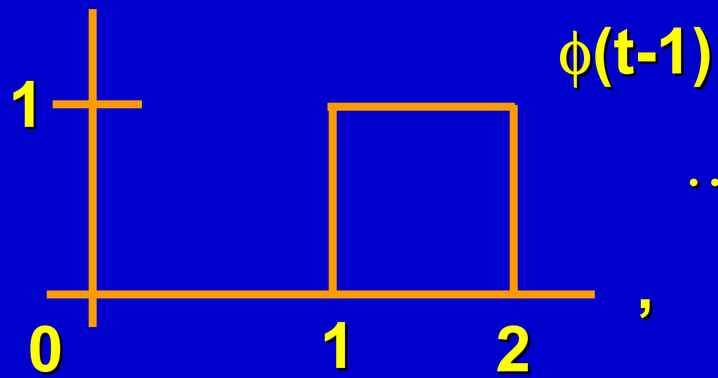

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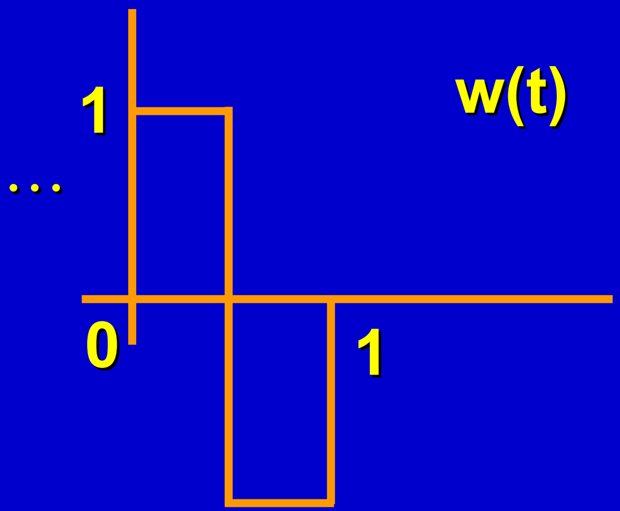
plus a combination of



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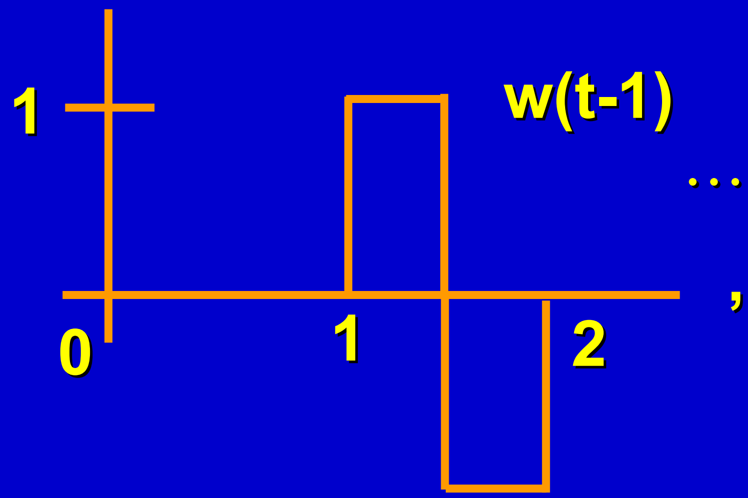

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Easy to see because

$$\phi(2t) = \tfrac{1}{2}[\phi(t) + w(t)]$$
$$\phi(2t - 1) = \tfrac{1}{2}[\phi(t) - w(t)]$$

- Suppose that f(t) is a function in $L^2(R)$. What are the coefficients, $a_j[k]$, of the projection of f(t) on to V_j? Call the projection $f_j(t)$,

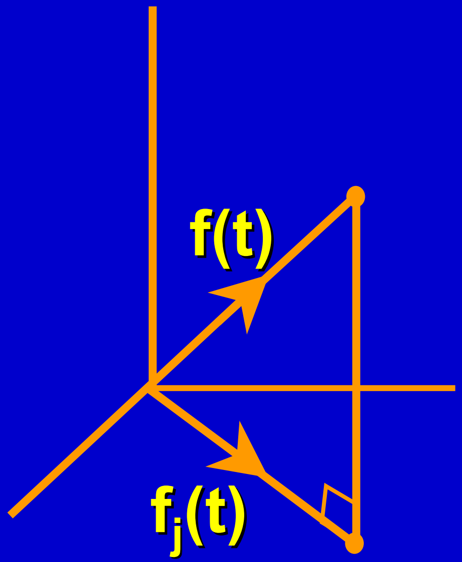
$$f_j(t) = \sum_k a_j[k]\phi_{j,k}(t)$$

$a_j[k]$ must minimize the distance between f(t) and $f_j(t)$

$$\frac{\partial}{\partial a_j[k]} \int_{-\infty}^{\infty} \{f(t) - f_j(t)\}^2 \, dt = 0$$

$$\int_{-\infty}^{\infty} 2\,\{f(t) - \sum_l a_j[l]\phi_{j,l}(t)\}\, \phi_{j,k}(t)dt = 0$$

$$a_j[k] = \int f(t)\phi_{j,k}(t)dt$$



- How does $\phi_{j,k}(t)$ relate to $\phi_{j-1,k}(t)$, $w_{j-1,k}(t)$?

$$\phi(t) = 2\sum_{\ell=0}^{N} h_0[\ell]\phi(2t - \ell)$$ refinement equation

$$\phi_{j-1,k}(t) = 2^{(j-1)/2}\,\phi(2^{j-1}t-k)$$

$$= 2^{(j-1)/2}.\ 2\sum_{\ell=0}^{N} h_0[\ell]\phi\,(2^{j}t - 2k - \ell)$$

$$\phi_{j-1,k}(t) = \sqrt{2}\sum_{\ell=0}^{N} h_0[\ell]\,\phi_{j,2k+\ell}(t)$$

Similarly, using the wavelet equation, we have

$$w_{j-1,k}(t) = \sqrt{2}\sum_{\ell=0}^{N} h_1[\ell]\phi_{j,2k+\ell}(t)$$

Multiresolution decomposition equations

$$a_{j-1}[n] = \int_{-\infty}^{\infty} f(t)\phi_{j-1,n}(t)\ dt$$

$$= \sqrt{2} \sum_{\ell} h_0[\ell] \int_{-\infty}^{\infty} f(t)\phi_{j,2n+\ell}(t)\ dt$$

$$= \sqrt{2} \sum_{\ell} h_0[\ell]\ a_j[2n+\ell]$$

So

$$a_{j-1}[n] = \sqrt{2} \sum_{k} h_0[k-2n]a_j[k]$$

$\rightarrow$ Convolution with $h_0[-n]$ followed by downsampling

Similarly

$$b_{j-1}[n] = \int_{-\infty}^{\infty} f(t)\, w_{j-1,n}(t)\, dt$$

which leads to

$$b_{j-1}[n] = \sqrt{2} \sum_{k} h_1[k - 2n]\, a_j[k]$$

Multiresolution reconstruction equation

Start with

$$f_j(t) = f_{j-1}(t) + g_{j-1}(t)$$

Multiply by $\phi_{j,n}(t)$ and integrate

$$\int_{-\infty}^{\infty} f_j(t)\, \phi_{j,n}(t)\, dt = \int_{-\infty}^{\infty} f_{j-1}(t)\phi_{j,n}(t)dt + \int_{-\infty}^{\infty} g_{j-1}(t)\phi_{j,n}(t)\, dt$$

So

$$a_j[n] = \sum_k a_{j-1}[k] \int_{-\infty}^{\infty} \phi_{j-1,k}(t)\, \phi_{j,n}(t)\, dt + \sum_k b_{j-1}[k] \int_{-\infty}^{\infty} w_{j-1,k}(t)\, \phi_{j,n}(t)\, dt$$

$$\int_{-\infty}^{\infty} \phi_{j-1,k}(t)\, \phi_{j,n}(t)\, dt = \sqrt{2} \sum_{\ell} h_0[\ell] \int_{-\infty}^{\infty} \phi_{j,2k+\ell}(t)\, \phi_{j,n}(t)\, dt$$

$$= \sqrt{2} \sum_{\ell} h_0[\ell]\, \delta[2k + \ell - n]$$

$$= \sqrt{2}\, h_0[n - 2k]$$

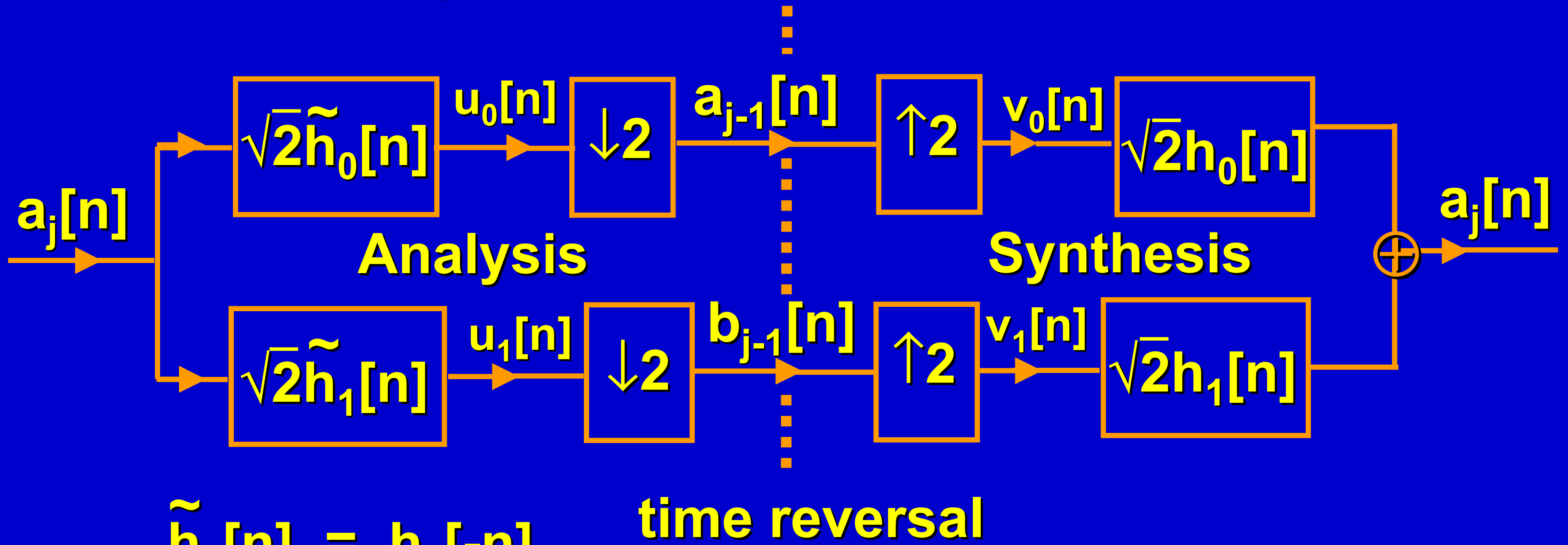
Similarly

$$\int_{-\infty}^{\infty} w_{j-1,k}(t)\phi_{j,n}(t)\, dt = \sqrt{2}\, h_1[n - 2k]$$

Result:

$$a_j[n] = \sqrt{2} \sum_{k} a_{j-1}[k] h_0[n - 2k] + \sqrt{2} \sum_{k} b_{j-1}[k] h_1[n - 2k]$$

Filter Bank Representation



$$\tilde{h}_0[n] = h_0[-n]$$

$$\tilde{h}_1[n] = h_1[-n]$$

Verify that filter bank implements MRA equations:

$$u_0[n] = \sqrt{2} \sum_k \tilde{h}_0[n - k]a_j[k]$$

$$= \sqrt{2} \sum_k h_0[k - n]a_j[k]$$

$$a_{j-1}[n] = u_0[2n] \qquad \text{downsample by 2}$$

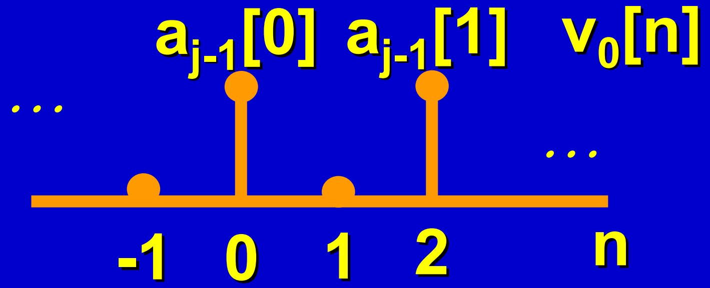
$$= \sqrt{2}\sum_k h_0[k-2n]a_j[k]$$

$$b_{j-1}[n] = u_1[2n]$$

$$= \sqrt{2}\sum_k h_1[k-2n]a_j[k]$$

$$a_j[n] = \sqrt{2}\sum_\ell h_0[n-\ell]v_0[\ell] + \sqrt{2}\sum_\ell h_1[n-\ell]v_1[\ell]$$

$$v_0[\ell] = \begin{cases} a_{j-1}[\ell/2] & ;\ \ell \text{ even} \\ 0 & ;\ \text{otherwise} \end{cases}$$



upsample by 2

So

$$a_j[n] = \sqrt{2}\sum_{\ell \text{ even}} h_0[n-\ell]a_{j-1}[\ell/2] + \sqrt{2}\sum h_1[n-\ell]b_{j-1}[\ell/2]$$

$$= \sqrt{2}\sum_k h_0[n-2k]a_{j-1}[k] + \sqrt{2}\sum_k h_1[n-2k]b_{j-1}[k]$$