

Course 18.327 and 1.130

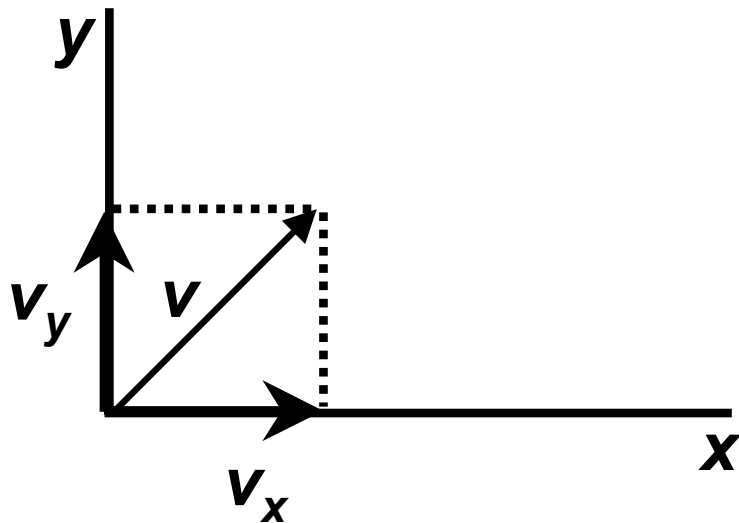
Wavelets and Filter Banks

Orthogonal wavelet bases: connection to orthogonal filters; orthogonality in the frequency domain. Biorthogonal wavelet bases.

Orthogonal Wavelets

2D Vector Space:

Basis vectors are i, j → orthonormal basis



v_x and v_y are the projections of v onto the x and y axes:

$$v_x = \langle v, i \rangle i$$

$$v_y = \langle v, j \rangle j$$

$$\langle \mathbf{v}, \mathbf{i} \rangle = [\mathbf{v}_x \quad \mathbf{v}_y] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{v}_x \quad \text{Inner Product}$$

Orthogonal multiresolution spaces:

V_j has an orthonormal basis $\underbrace{\{2^{j/2} \phi(2^j t - k) : -\infty \leq k \leq \infty\}}_{\phi_{j,k}(t)}$

W_j has an orthonormal basis $\underbrace{\{2^{j/2} w(2^j t - k) : -\infty \leq k \leq \infty\}}_{w_{j,k}(t)}$

Orthonormal means

$$\langle \phi_{j,k}(t), \phi_{j,l}(t) \rangle = \int_{-\infty}^{\infty} 2^{j/2} \phi(2^j t - k) 2^{j/2} \phi(2^j t - l) dt = \delta[k - l]$$

$$\langle w_{j,k}(t), w_{j,l}(t) \rangle = \int_{-\infty}^{\infty} 2^{j/2} w(2^j t - k) 2^{j/2} w(2^j t - l) dt = \delta[k - l]$$

For orthogonal multiresolution spaces, we have

$$V_j \perp W_j$$

So

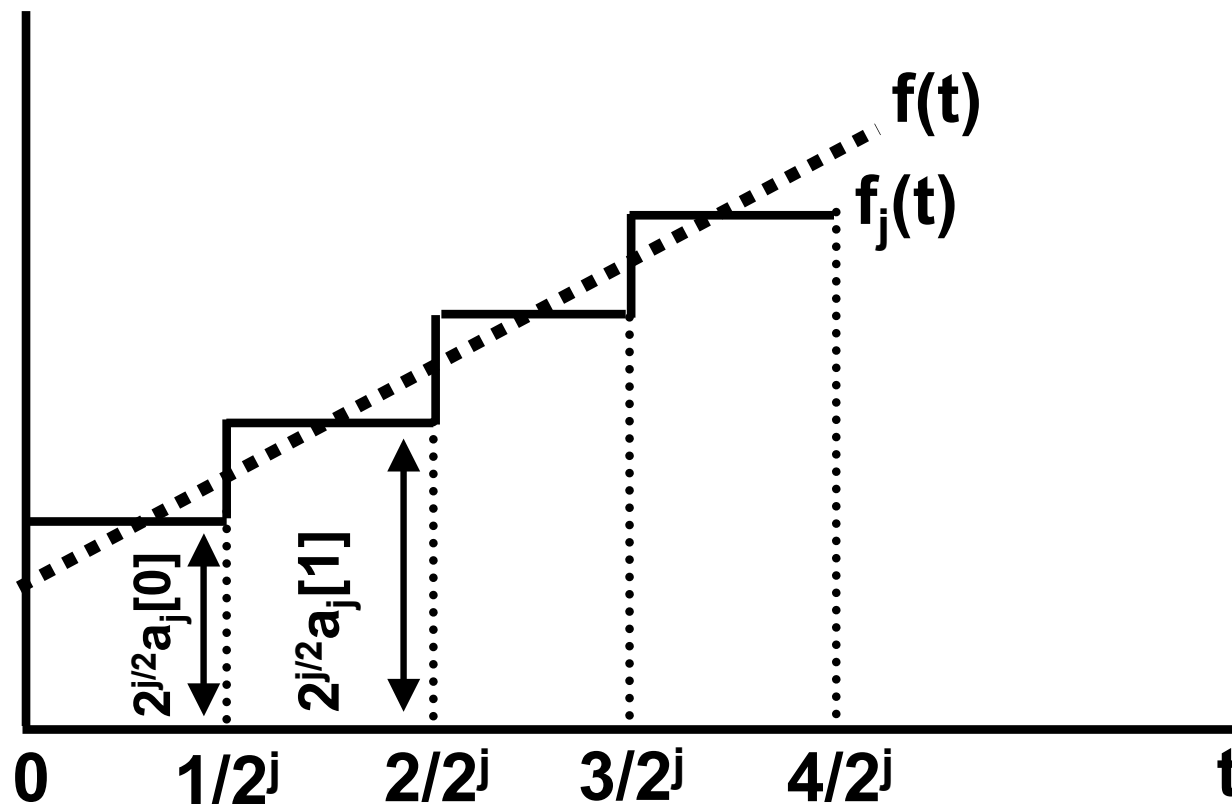
$$\langle \phi_{j,k}(t), w_{j,l}(t) \rangle = 0$$

Projection of an L^2 function, $f(t)$, onto V_j :

with
$$f_j(t) = \sum_k a_j[k] \phi_{j,k}(t)$$

$$a_j[k] = \langle f(t), \phi_{j,k}(t) \rangle$$

Haar example



Projection of $f(t)$ onto W_j :

$$\mathbf{g}_j(t) = \sum_k \mathbf{b}_j[k] \mathbf{w}_{j,k}(t)$$

with

$$\mathbf{b}_j[k] = \langle \mathbf{f}(t), \mathbf{w}_{j,k}(t) \rangle$$

Biorthogonal Wavelet Bases

Two scaling functions and two wavelets:
Synthesis:

$$\phi(t) = 2 \sum_k f_0[k] \phi(2t - k)$$

$$w(t) = 2 \sum_k f_1[k] \phi(2t - k)$$

Analysis:

$$\tilde{\phi}(t) = 2 \sum_k h_0[-k] \tilde{\phi}(2t - k)$$

$$\tilde{w}(t) = 2 \sum_k h_1[-k] \tilde{\phi}(2t - k)$$

Two sets of multiresolution spaces:

$$\{0\} \subset \dots \subset V_0 \subset V_1 \subset \dots \subset V_j \subset \dots \subset L^2(\square)$$

$$\{0\} \subset \dots \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \dots \subset \tilde{V}_j \subset \dots \subset L^2(\square)$$

$$V_j + W_j = V_{j+1}$$

$$\tilde{V}_j + \tilde{W}_j = \tilde{V}_{j+1}$$

Spaces are orthogonal w.r.t. each other i.e.

$$V_j \perp \tilde{W}_j \quad \tilde{V}_j \perp W_j$$

V_0 has a basis $\{\phi(t - k) : -\infty < k < \infty\}$

\tilde{V}_0 has a basis $\{\tilde{\phi}(t - k) : -\infty < k < \infty\}$

W_0 has a basis $\{w(t - k) : -\infty < k < \infty\}$

\tilde{W}_0 has a basis $\{\tilde{w}(t - k) : -\infty < k < \infty\}$

Bases are orthogonal w.r.t. each other i.e.

$$\int \phi(t) \tilde{\phi}(t-k) dt = \delta[k] \quad \int \phi(t) \tilde{w}(t-k) dt = 0$$

$$\int w(t) \tilde{\phi}(t-k) dt = 0 \quad \int w(t) \tilde{w}(t-k) dt = \delta[k]$$

Equivalent to perfect reconstruction conditions on filters

Representation of functions in a biorthogonal basis:

$$f(t) = \sum_k c_k \phi(t-k) + \sum_{j=0}^{\infty} \sum_k d_{j,k} 2^{j/2} w(2^j t - k)$$

$$c_k = \int f(t) \tilde{\phi}(t-k) dt$$

$$d_{j,k} = 2^{j/2} \int f(t) \tilde{w}(2^j t - k) dt$$

Similarly, we can represent $f(t)$ in the dual basis

$$f(t) = \sum_k \tilde{c}_k \tilde{\phi}(t - k) + \sum_{j=0}^{\infty} \sum_k \tilde{d}_{j,k} 2^{j/2} \tilde{w}(2^j t - k)$$

$$\tilde{c}_k = \int f(t) \phi(t - k) dt$$

$$\tilde{d}_{j,k} = 2^{j/2} \int f(t) w(2^j t - k) dt$$

Note: When $f_0[k] = h_0[-k]$ and $f_1[k] = h_1[-k]$, we have

$$\phi(t) = \tilde{\phi}(t) \Rightarrow V_j = \tilde{V}_j$$

$$w(t) = \tilde{w}(t) \Rightarrow W_j = \tilde{W}_j$$

i.e. we have orthogonal wavelets!

Connection between orthogonal wavelets and orthogonal filters

Start with the orthonormality requirement on scaling functions

$$\delta[n] = \int_{-\infty}^{\infty} \phi(t) \phi(t - n) dt$$

And then change scale using the refinement equation:

$$\begin{aligned} \delta[n] &= \int_{-\infty}^{\infty} 2 \sum_k h_0[k] \phi(2t - k) 2 \sum_l h_0[l] \phi(2(t - n) - l) dt \\ &= 4 \sum_k h_0[k] \sum_l h_0[l] \int_{-\infty}^{\infty} \phi(\tau) \phi(\tau + k - 2n - l) d\tau / 2 \\ &= 2 \sum_k h_0[k] \sum_l h_0[l] \delta[-k + 2n + l] \end{aligned}$$

$$\delta[n] = 2 \sum_k h_0[k] h_0[k - 2n]$$

This is the “double shift” orthogonality condition that we previously encountered for orthogonal filters.

So orthogonal filters are necessary for orthogonal wavelets. Are they sufficient?

Consider the cascade algorithm

$$\phi^{(i+1)}(t) = 2 \sum_{k=0}^N h_0[k] \phi^{(i)}(2t - k)$$

If the filters are orthogonal and

$$\int_{-\infty}^{\infty} \phi^{(i)}(t) \phi^{(i)}(t - n) dt = \delta[n]$$

then

$$\int_{-\infty}^{\infty} \phi^{(i+1)}(t) \phi^{(i+1)}(t - n) dt = \delta[n]$$

Our initial guess for the iteration was $\phi^{(0)}(t) = \text{box on } [0,1]$ which is orthonormal w.r.t. its shifts. By induction, we have $\int_{-\infty}^{\infty} \phi(t) \phi(t - n) dt = \delta[n]$ as the limit, provided that the cascade algorithm converges.

Note: with the alternating flip requirement, which was necessary for the highpass filter in the case of orthogonal filters, we can show that

$$\int_{-\infty}^{\infty} w(t) w(t - n) dt = \delta[n]$$

and

$$\int_{-\infty}^{\infty} \phi(t) w(t - n) dt = 0$$

Orthogonality in the Frequency Domain

Let

$$a[n] = \int_{-\infty}^{\infty} \phi(t) \phi(t - n) dt$$

Use Parseval's theorem

$$a[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\Omega) \overline{\hat{\phi}(\Omega)} e^{-i\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\phi}(\Omega)|^2 e^{i\Omega n} d\Omega$$

Trick: split integral over entire Ω axis into a sum of integrals over 2π intervals

$$\begin{aligned} a[n] &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} |\hat{\phi}(\Omega + 2\pi k)|^2 e^{i(\Omega + 2\pi k)n} d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi k)|^2 e^{i\Omega n} d\pi \end{aligned}$$

Take the Discrete Time Fourier Transform of both sides

$$A(\omega) \equiv \sum_n a[n]e^{-i\omega n} = \sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$

If $\phi(t - n)$ are orthonormal, then $a[n] = \delta[n]$

$$\Rightarrow A(\omega) = 1$$

So the scaling function and its shifts are orthogonal if

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = 1$$

Note: if we set $\omega = 0$, then

$$\sum_k |\hat{\phi}(2\pi k)|^2 = 1$$

and since $\hat{\phi}(0) = 1$, we see that

$$\hat{\phi}(2\pi k) = 0 \text{ for } k \neq 0$$

Connection between orthogonal wavelets and orthogonal filters (in frequency domain):

Start with an orthogonal scaling function:

$$1 = \sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$

and then change scale using the refinement equation in the frequency domain:

$$\hat{\phi}(\Omega) = H_0(\Omega/2) \hat{\phi}(\Omega/2)$$

$$1 = \sum_{k=-\infty}^{\infty} |H_0(\Omega/2 + \pi k)|^2 |\hat{\phi}(\Omega/2 + \pi k)|^2$$

Trick: split sum into even and odd

$$\begin{aligned} 1 &= \sum_{k=-\infty}^{\infty} |H_0(\Omega/2 + 2\pi k)|^2 |\hat{\phi}(\Omega/2 + 2\pi k)|^2 + \\ &\quad \sum_{k=-\infty}^{\infty} |H_0(\Omega/2 + \pi(2k + 1))|^2 |\hat{\phi}(\Omega/2 + \pi(2k + 1))|^2 \\ &= |H_0(\Omega/2)|^2 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega/2 + 2\pi k)|^2 + \\ &\quad |H_0(\Omega/2 + \pi)|^2 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega/2 + \pi + 2\pi k)|^2 \end{aligned}$$

But each of the two infinite sums is equal to 1

So the discrete time filter $H_0(\omega)$ must be orthogonal:

$$1 = |H_0(\omega)|^2 + |H_0(\omega + \pi)|^2$$