Course 18.327 and 1.130
Wavelets and Filter Banks

Signal and Image Processing: finite length signals; boundary filters and boundary wavelets; wavelet compression algorithms.

Finite-Length Signals

$$x[n] \longrightarrow \boxed{H(\omega)} \longrightarrow y[n]$$

$$\begin{bmatrix} y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \\ y_{N-1} \\ \vdots \end{bmatrix} = \begin{bmatrix} h_0 & & & & & \\ h_1 & h_0 & h_{-1} & & & \\ h_2 & h_1 & h_0 & h_{-1} & & \\ & & h_1 & h_0 & h_{-1} & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{bmatrix} \begin{bmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \\ x_{N-1} \\ \vdots \end{bmatrix}$$

unknown

length N
(finite length)

unknown

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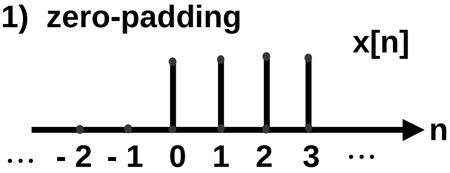
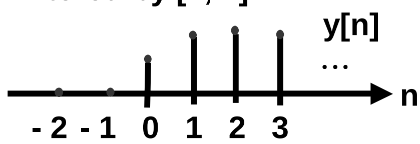
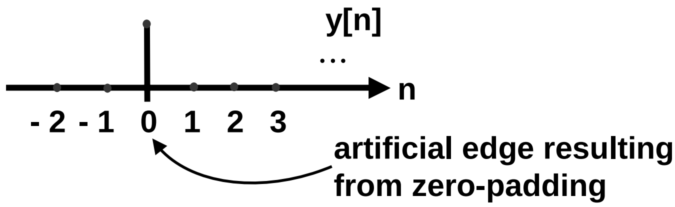
1) zero-padding



filtered by [1, 1]

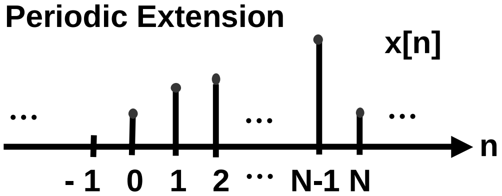


filtered by [1, -1]



2) Periodic Extension

$x[N] = x[0]$



wrap-around

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} h_1 & h_0 & h_{-1} & h_{-2} & \cdots & h_2 & h_1 \\ & h_1 & h_0 & h_{-1} & \cdots & h_3 & h_2 \\ & & & & & & \\ h_{-1} & h_{-2} & & \cdots & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} \vdots \\ x_{N-2} \\ x_{N-1} \\ x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_0 \\ \vdots \end{bmatrix}$$

N-output circulant matrix = H

What is the eigenvector for the circulant matrix?

$$[\,1 \;\; e^{i\omega} \;\; e^{i2\omega} \;\; \cdots \;\; e^{i(N-1)\omega}\,]^T$$

We need

$$e^{iN\omega} = 1 = e^{i0\omega}$$

$$\therefore \;\; N\omega = 2\pi k \quad , \qquad \boxed{\omega = \frac{2\pi k}{N}}$$

discrete set of ω's

For the 0^{th} row,

$$H[k] = \sum_{n=0}^{N-1} h[n]\, e^{-i\frac{2\pi k}{N}n}$$

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$$[\,H\,] \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^{N-1} \\ 1 & w^2 & w^4 & & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & & & \end{bmatrix} = [F] \begin{bmatrix} H[0] & & & \\ & H[1] & & \\ & & \ddots & \\ & & & H[N-1] \end{bmatrix}$$

$$\underbrace{\uparrow_{k=0} \;\; \uparrow_{k=1} \qquad \qquad \uparrow_{k=N-1}}_{F} \qquad w = e^{i\frac{2\pi}{N}}$$

$HF = F\Lambda$ $\qquad$ Λ contains the Fourier coefficients

$$H[k] = \sum_n h[n] e^{-i\frac{2\pi k}{N}n}$$

$$\sum_n \sum_\ell h[n-\ell]x[\ell]e^{-i\frac{2\pi k}{N}n} = H[k]X[k]$$

If $x[\ell] = e^{i\frac{2\pi k_0}{N}\ell}$ $\quad \Rightarrow X[k] = \delta[k-k_0]$

$\Rightarrow H[k]X[k] = H[k_0]X[k]$

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3) Symmetric Extension
   1) Whole point symmetry – when filter is whole point symmetric.
   2) Half point symmetry – when filter is half point symmetric.

e.g. Whole point symmetry: filter and signal

$$\begin{bmatrix} h_1x_2 + h_0x_1 + h_1x_0 \\ h_1x_1 + h_0x_0 + h_1x_1 \\ h_1x_0 + h_0x_1 + h_1x_2 \\ \\ \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & & & \\ h_1 & h_0 & h_1 & & \\ & h_1 & h_0 & h_1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

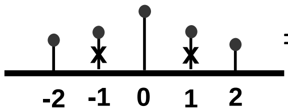
e.g. whole point symmetry – filter,
half-point symmetry - signal

$$\begin{bmatrix} h_1x_2 + h_0x_1 + h_1x_0 \\ h_1x_1 + h_0x_0 + h_1x_0 \\ \hdashline h_1x_0 + h_0x_0 + h_1x_1 \\ h_1x_0 + h_0x_1 + h_1x_2 \\ \\ \end{bmatrix} = \begin{bmatrix} h_1 & h_0 & h_1 & & & \\ & h_1 & h_0 & h_1 & & \\ & & h_1 & h_0 & h_1 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \\ \hdashline x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

Half point symmetry — Whole point symmetry

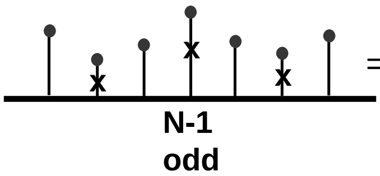
Downsampling a whole-point symmetric signal with even length N

at the left boundary:



$\Rightarrow$ **still whole-point symmetric after $\downarrow 2$.**

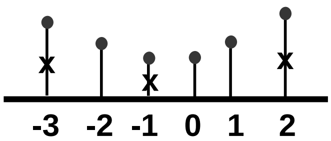
at the right boundary:



$\Rightarrow$ **half-point symmetric after $\downarrow 2$.**

E.g. 9/7 filter: whole-point symmetric
use the above extension for signal $\Rightarrow$ **N** — **N/2**, **N/2** **exactly**

Downsample a half-point symmetric signal



$\Rightarrow$ **nothing guaranteed**

Linear-phase filters

$$H(\omega) = A(\omega)e^{-i\omega\alpha}$$

1) **half-point symmetric,** α = **fraction**

2) **whole-point symmetric,** α = **integer**

Symmetric extension of finite-length signal

$$X(\omega) = B(\omega)e^{-i\omega\beta}$$

The output:

$Y(\omega) = H(\omega)X(\omega)$

$Y(\omega)$	$H(\omega)$	$X(\omega)$
W	W	W
W	H	H
H	H	W
H	W	H

W = whole-point symmetry
H = half-point symmetry

The above extensions ensure the continuity of function values at boundaries, but not the continuity of derivatives at boundaries.

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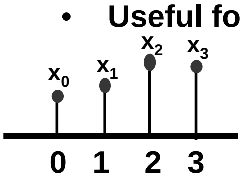
4) Polynomial Extrapolation (not useful in image processing)

- Useful for PDE with boundary conditions.



4 coefficients $\Rightarrow$ fits up to 3rd order polynomials.

$$a + bn + cn^2 + dn^3 = x(n)$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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Then,

$$x_{-1} = [1\ \ -1\ \ 1\ \ -1] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = [1\ \ -1\ \ 1\ \ -1]\ A^{-1} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

PDE

$$f(x) = \sum_k c_k \phi(x - k)$$

Assume f(x) has polynomial behavior near boundaries

$$\sum_{i=0}^{p-1} \alpha_i x^i = f(x) = \sum_k c_k \phi(x - k)$$

$\{\phi\ (\bullet - k)\}$ orthonormal

$$\Rightarrow \sum_{i=0}^{p-1} \alpha_i \underbrace{\int \phi(x - k) x^i dx}_{\mu_k^i} = c_k$$

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$$\begin{bmatrix} \mu_0^0 & \mu_0^1 & \cdots & \mu_0^{p-1} \\ \mu_1^0 & \mu_1^1 & \mu_1^2 & \cdots \\ \vdots & & & \\ & & & \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{p-1} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{bmatrix}$$

Using the computed α_i's, we can extrapolate,

e.g. $c_{-1} = [\mu_{-1}^0\ \ \mu_{-1}^1\ \cdots\ \mu_{-1}^{p-1}] \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{p-1} \end{bmatrix}$

DCT idea of symmetric extension

cf. DFT $X[k] = \sum_n x[n] e^{-i \frac{2\pi k}{N} n}$

complex-valued

Want real-valued results.

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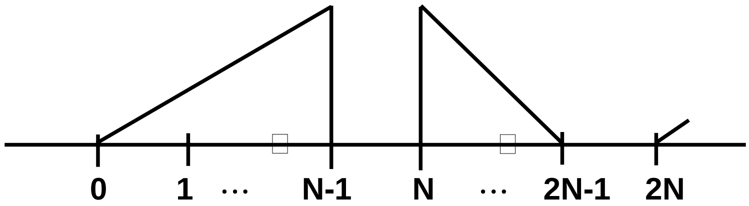

DFT of this extended signal:

$$\sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi k}{2N}n} + \underbrace{\sum_{n=N}^{2N-1} x[2N-1-n]e^{-i\frac{2\pi k}{2N}n}}_{\square}$$

$$\sum_{m=0}^{N-1} x[m]e^{-i\frac{2\pi k}{2N}(2N-1-m)}$$

$$= \sum_{n=0}^{N-1} x[n]\left\{e^{-i\frac{2\pi k}{2N}n} + e^{-i\frac{2\pi k}{2N}(2N-1-n)}\right\}$$

$$X(k) \square\ c_k \sum_{n=0}^{N-1} \sqrt{\tfrac{2}{N}}\, x[n]\cos\frac{\pi k}{N}(n+\tfrac{1}{2})$$ ··· **DCT – II used in JPEG**

$$c_k = \begin{cases} 1/\sqrt{2} & k = 0 \\ 1 & k = 1, 2, \ldots, N-1 \end{cases}$$

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