Course 18.327 and 1.130
Wavelets and Filter Banks

Smoothness of wavelet bases: Convergence of the cascade algorithm (Condition E); Splines. Bases vs. frames.

Smoothness of Wavelet Bases

Use eigenvalue analysis to study convergence of the cascade algorithm and smoothness of resulting scaling function.

The cascade algorithm revisited:

$$\phi^{(i+1)}(t) = 2\sum_{k} h_0[k]\, \phi^{(i)}(2t - k)$$

Consider the behavior of the inner products

$$a^{(i)}[n] = \int_{-\infty}^{\infty} \phi^{(i)}(t)\, \phi^{(i)}(t + n)\, dt$$

as $i \rightarrow \infty$ to understand convergence.

$$a^{(i+1)}[n] = \int_{-\infty}^{\infty}\{2\sum_k h_0[k]\phi^{(i)}(2t - k)\}\{2\sum_\ell h_0[\ell]\phi^{(i)}(2t + 2n - \ell)\}dt$$

$$= 2\sum_\ell h_0[\ell] \sum_k h_0[k]\, a^{(i)}[\underbrace{k + 2n - \ell}_{m}]$$

$$= 2\sum_\ell h_0[\ell] \sum_m h_0[m \underbrace{- 2n + \ell}_{-r}]\, a^{(i)}[m]$$

$$= 2\sum_r \underbrace{h_0[2n - r]}_{\text{Filter with } h_0[n] \text{ and then downsample}} \sum_m \underbrace{h_0[-(r - m)]}_{\text{Filter with } h_0[-n]} a^{(i)}[m]$$

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In matrix form:

$$\underline{a}^{(i+1)} = \underbrace{(\downarrow 2)\, 2\, H_0 H_0^T}_{T}\, \underline{a}^{(i)} \quad ; \qquad H_0 \rightarrow \text{Toeplitz matrix}$$

Iteration converges if the eigenvalues of the transition matrix T satisfy

$$|\lambda| \leq 1$$

with only a simple eigenvalue at $\lambda = 1$.

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Splines

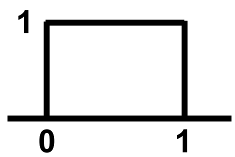
Splines are scaling functions whose filters only have zeros at π i.e.

$$H_0(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^p$$

$$h_0[n] = \frac{1}{2^p}\binom{p}{n} \quad ; \quad n = 0, 1, \ldots, p$$

binomial coefficients

Consider p = 1

$\phi(t) =$ 

$$\hat{\phi}(\Omega) = e^{-i\Omega/2} \frac{\sin \Omega/2}{\Omega/2}$$

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What happens when p = 2 ?

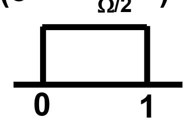
$$H_0(\omega) = \underbrace{\left(\frac{1 + e^{-i\omega}}{2}\right)}_{H_0^1(\omega)}\underbrace{\left(\frac{1 + e^{-i\omega}}{2}\right)}_{H_0^2(\omega)}$$

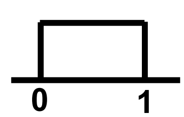
$$\hat{\phi}(\Omega) = \prod_{j=1}^{\infty} H_0(\Omega/2^j)$$

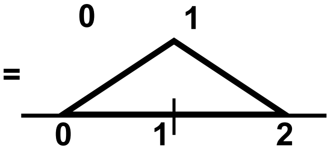
$$= \prod_{j=1}^{\infty} H_0^1(\Omega/2^j) \cdot \prod_{j=1}^{\infty} H_0^2(\Omega/2^j)$$

$$= \hat{\phi}^1(\Omega) \cdot \hat{\phi}^2(\Omega)$$

$$= \left(e^{-i\Omega/2} \frac{\sin \Omega/2}{\Omega/2}\right)^2$$

$\phi(t) =$ 

$*$ 

$=$ 

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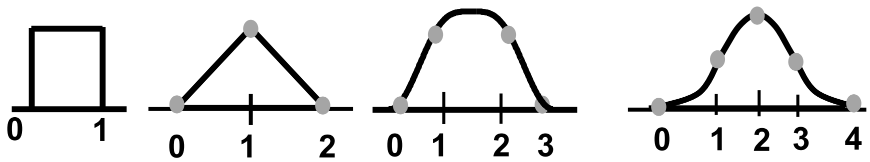
More generally

$$\phi(\Omega) = \left(e^{-i\Omega/2}\,\frac{\sin \Omega/2}{\Omega/2}\right)^p$$

$$\phi(t) = \phi_{box}(t) \ast \phi_{box}(t) \ast \ldots \ast \phi_{box}(t) \qquad \text{(p terms)}$$



p = 1 p = 2 p = 3 p = 4

$\phi(t)$ is piecewise polynomial of degree $p - 1$. The derivatives, $\phi^{(s)}(t)$, exist for $s \leq p - 1$ and they are continuous for $s \leq p - 2$.

e.g. Cubic spline ($p = 4$) is C^2 continuous.

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Alternatively, measure smoothness in L^2 sense:

$$||\hat{\phi}^{(s)}(t)||^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| (i\Omega)^s\, \hat{\phi}(\Omega) \right|^2 d\Omega \qquad \text{(by Plancherel)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega^{2s}\, \frac{4^p |\sin \Omega/2|^{2p}}{\Omega^{2p}}\, d\Omega$$

$< \infty$ when $2s - 2p < -1$

Note: $\int_{-\infty}^{\infty} \frac{1}{\Omega}\, d\Omega$ is limiting case

So, $\phi(t)$ has s derivatives in the L^2 sense for all $s < s_{max}$, where

$$\boxed{s_{max} = p - ½}$$

Valid for splines

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Non-spline Scaling Functions
In general, we have

$$H_0(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^p Q(\omega)$$

so that

$$\phi(t) = \underbrace{\phi_p(t)}_{\text{pth order spline}} \ast \underbrace{\phi_q(t)}_{\text{ugly}}$$

Notice that the approximation power of $\phi(t)$ comes entirely from $\phi_p(t)$:

Suppose that we write

$$\sum_k c_k \phi_p(t - k) = t^\ell$$

for some ℓ $(0 \le \ell < p)$.

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Then we have

$$\begin{aligned}\sum_k c_k \, \phi(t - k) &= \phi_q(t) \ast t^\ell \\ &= \int_{-\infty}^{\infty} \phi_q(\tau)(t - \tau)^\ell \, d\tau \\ &= \sum_{i=0}^{\ell} \underbrace{\binom{\ell}{i} \int_{-\infty}^{\infty} \phi_q(\tau)(-\tau)^{\ell-i} \, d\tau}_{\alpha_i} \, . \, t^i \\ &= \text{polynomial of degree } \ell.\end{aligned}$$

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What about smoothness (in L^2 sense)?
Smoothness is given by

$$\boxed{s_{max} = p - \tfrac{1}{2} \log_2 |\lambda_{max}(T_Q)|}$$

where

$$T_Q = (\downarrow 2) 2QQ^T \quad \text{Transition matrix for } Q(\omega)$$

Alternatively, look at the transition matrix for $H_0(\omega)$,

$$T = (\downarrow 2) 2H_0H_0^T$$

T has 2p special eigenvalues due to the zeros at π:

$$\lambda = 1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, \left(\tfrac{1}{2}\right)^{2p-1}$$

Disregard these eigenvalues and look at the largest non-special eigenvalue, λ_{max}. Then the smoothness is given by

$$s_{max} = -\tfrac{1}{2} \log_2 |\lambda_{max}(T)|$$

$$\lambda_{max}(T) = 4^{-s_{max}}$$

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