Appendix B

Finite difference methods for wave equations

Many types of numerical methods exist for computing solutions to wave equations – finite differences are the simplest, though often not the most accurate ones.

Consider for illustration the 1D time-dependent problem

$$m(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad x \in [0,1],$$

with smooth $f(x,t)$, and, say, zero initial conditions. The simplest finite difference scheme for this equation is set up as follows:

- Space is discretized over $N+1$ points as $x_j = j\Delta x$ with $\Delta x = \frac{1}{N}$ and $j = 0, \ldots, N$.

- Time is discretized as $t_n = n\Delta t$ with $n = 0, 1, 2, \ldots$. Call u_j^n the computed approximation to $u(x_j, t_n)$. (In this appendix, n is a superscript.)

- The centered finite difference formula for the second-order spatial derivative is
$$\frac{\partial^2 u}{\partial x^2}(x_j, t_n) = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + O((\Delta x)^2),$$
provided u is sufficiently smooth – the $O(\cdot)$ notation hides a multiplicative constant proportional to $\partial^4 u/\partial x^4$.

- Similarly, the centered finite difference formula for the second-order time derivative is

$$\frac{\partial^2 u}{\partial t^2}(x_j, t_n) = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} + O((\Delta t)^2),$$

  provided u is sufficiently smooth.

- Multiplication by $m(x)$ is realized by multiplication on the grid by $m(x_j)$. Gather all the discrete operators to get the discrete wave equation.

- The wave equation is then solved by marching: assume that the values of u_j^{n-1} and u_j^n are known for all j, then isolate u_j^{n+1} in the expression of the discrete wave equation.

Dirichlet boundary conditions are implemented by fixing. e.g., $u_0 = a$. Neumann conditions involve a finite difference, such as $\frac{u_1 - u_0}{\Delta x} = a$. The more accurate, centered difference approximation $\frac{u_1 - u_{-1}}{2\Delta x} = a$ with a ghost node at u_{-1} can also be used, provided the discrete wave equation is evaluated one more time at x_0 to close the resulting system. In 1D the absorbing boundary condition has the explicit form $\frac{1}{c}\partial_t u \pm \partial_x u = 0$ for left (-) and right-going (+) waves respectively, and can be implemented with adequate differences (such as upwind in space and forward in time).

The grid spacing Δx is typically chosen as a small fraction of the representative wavelength in the solution. The time step Δt is limited by the CFL condition $\Delta t \leq \Delta x \,/\, \max_x c(x)$, and is typically taken to be a fraction thereof.

In two spatial dimensions, the simplest discrete Laplacian is the 5-point stencil which combines the two 3-point centered schemes in x and in y. Its accuracy is also $O(\max\{\Delta x)^2, (\Delta y)^2\})$. Designing good absorbing boundary conditions is a somewhat difficult problem that has a long history. The currently most popular solution to this problem is to slightly expand the computational domain using an absorbing, perfectly-matched layer (PML).

More accurate schemes can be obtained from higher-order finite differences. Low-order schemes such as the one explained above typically suffer from unacceptable numerical dispersion at large times. If accuracy is a big concern, spectral methods (spectral elements, Chebyshev polynomials, etc.) are by far the best way to solve wave equations numerically with a controlled, small number of points per wavelength.