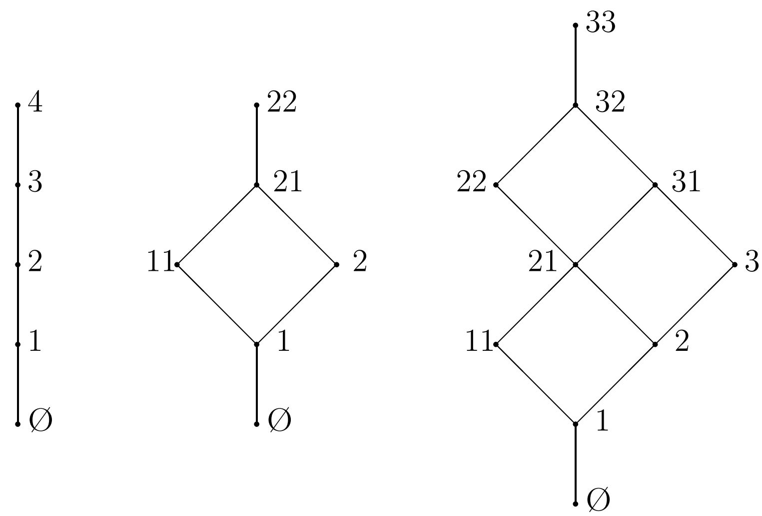
6 Young diagrams and q-binomial coefficients.

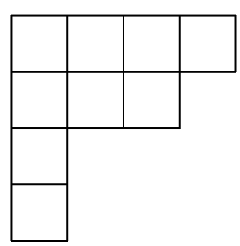
A *partition* λ of an integer $n \geq 0$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of integers $\lambda_i \geq 0$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum_{i\geq 1} \lambda_i = n$. Thus all but finitely many λ_i are equal to 0. Each $\lambda_i > 0$ is called a *part* of λ. We sometimes suppress 0's from the notation for λ, e.g., $(5,2,2,1)$, $(5,2,2,1,0,0,0)$, and $(5,2,2,1,0,0,\ldots)$ all represent the same partition λ (of 10, with four parts). If λ is a partition of n, then we denote this by $\lambda \vdash n$ or $|\lambda| = n$.

6.1 Example. There are seven partitions of 5, namely (writing e.g. 221 as short for $(2,2,1)$): 5, 41, 32, 311, 221, 2111, and 11111.

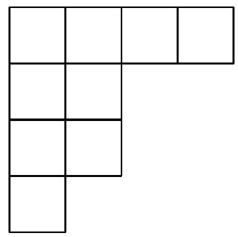
The subject of partitions of integers has been extensively developed, and we will only be concerned here with a small part related to our previous discussion. Given positive integers m and n, let $L(m,n)$ denote the set of all partitions with at most m parts and with largest part at most n. For instance, $L(2,3) = \{\emptyset, 1, 2, 3, 11, 21, 31, 22, 32, 33\}$. (Note that we are denoting by $\emptyset$ the unique partition $(0,0,\ldots)$ with no parts.) If $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ are partitions, then define $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all i. This makes the set of all partitions into a very interesting poset, denoted Y and called *Young's lattice* (named after the British mathematician Alfred Young, 1873–1940). (It is called "Young's lattice" rather than "Young's poset" because it turns out to have certain properties which define a *lattice*. However, these properties are irrelevant to us here, so we will not bother to define the notion of a lattice.) We will be looking at some properties of Y in Section 8. The partial ordering on Y, when restricted to $L(m,n)$, makes $L(m,n)$ into a poset which also has some fascinating properties. The diagrams below show $L(1,4)$, $L(2,2)$, and $L(2,3)$.



There is a nice geometric way of viewing partitions and the poset $L(m, n)$. The *Young diagram* (somtimes just called the *diagram*) of a partition λ is a left-justified array of squares, with λ_i squares in the ith row. For instance, the Young diagram of $(4, 3, 1, 1)$ looks like:

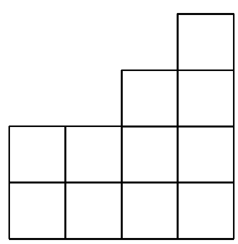


If dots are used instead of boxes, then the resulting diagram is called a *Ferrers diagram.* The advantage of Young diagrams over Ferrers diagrams is that we can put numbers in the boxes of a Young diagram, which we will do in Section 7. Observe that $L(m, n)$ is simply the set of Young diagrams D fitting in an $m \times n$ rectangle (where the upper-left (northwest) corner of D is the same as the northwest corner of the rectangle), ordered by inclusion. *We will always assume that when a Young diagram D is contained in a rectangle R, the northwest corners agree.* It is also clear from the Young diagram point of view that $L(m, n)$ and $L(n, m)$ are isomorphic partially ordered sets, the isomorphism being given by transposing the diagram (i.e., interchanging rows and columns). If λ has Young diagram D, then the partition whose diagram is D^t (the transpose of D) is called the *conjugate* of λ and is denoted λ'. For instance, $(4,3,1,1)' = (4,2,2,1)$, with diagram



6.2 Proposition. *$L(m,n)$ is graded of rank mn and rank-symmetric. The rank of a partition λ is just $|\lambda|$ (the sum of the parts of λ or the number of squares in its Young diagram).*

Proof. As in the proof of Proposition 5.6, we leave to the reader everything except rank-symmetry. To show rank-symmetry, consider the complement $\bar{\lambda}$ of λ in an $m \times n$ rectangle R, i.e., all the squares of R except for λ. (Note that $\bar{\lambda}$ depends on m and n, and not just λ.) For instance, in $L(4,5)$, the complement of $(4,3,1,1)$ looks like



If we rotate the diagram of $\bar{\lambda}$ by 180° then we obtain the diagram of a partition $\tilde{\lambda} \in L(m,n)$ satisfying $|\lambda| + |\tilde{\lambda}| = mn$. This correspondence between λ and $\tilde{\lambda}$ shows that $L(m,n)$ is rank-symmetric. $\square$

Our main goal in this section is to show that $L(m,n)$ is rank-unimodal and Sperner. Let us write $p_i(m,n)$ as short for $p_i(L(m,n))$, the number of elements of $L(m,n)$ of rank i. Equivalently, $p_i(m,n)$ is the number of partitions of i with largest part at most n and with at most m parts, or, in other words, the number of distinct Young diagrams with i squares which fit inside an $m \times n$ rectangle (with the same northwest corner, as explained previously). Though not really necessary for this goal, it is nonetheless interesting to obtain some information on these numbers $p_i(m, n)$. First let us consider the total number $|L(m, n)|$ of elements in $L(m, n)$.

6.3 Proposition. *We have* $|L(m,n)| = \binom{m+n}{m}$.

Proof. We will give an elegant combinatorial proof, based on the fact that $\binom{m+n}{m}$ is equal to the number of sequences $a_1, a_2, \ldots, a_{m+n}$, where each a_j is either N or E, and there are m N's (and hence n E's) in all. We will associate a Young diagram D contained in an $m \times n$ rectangle R with such a sequence as follows. Begin at the lower left-hand corner of R, and trace out the southeast boundary of D, ending at the upper right-hand corner of R. This is done by taking a sequence of unit steps (where each square of R is one unit in length), each step either north or east. Record the sequence of steps, using N for a step to the north and E for a step to the east.

Example. Let $m = 5$, $n = 6$, $\lambda = (4, 3, 1, 1)$. Then R and D are given by:

×	×	×	×		
×	×	×			
×					
×					

The corresponding sequence of N's and E's is $NENNEENENENEE$.

It is easy to see (left to the reader) that the above correspondence gives a bijection between Young diagrams D fitting in an $m \times n$ rectangle R, and sequences of m N's and n E's. Hence the number of diagrams is equal to $\binom{m+n}{m}$, the number of sequences. $\square$

We now consider how many elements of $L(m, n)$ have rank i. To this end, let q be an indeterminate; and given $j \geq 1$ define $[j] = 1 + q + q^2 + \cdots + q^{j-1}$. Thus $[1] = 1$, $[2] = 1 + q$, $[3] = 1 + q + q^2$, etc. Note that $[j]$ is a polynomial in q whose value at $q = 1$ is just j (denoted $[j]_{q=1} = j$). Next define $[j]! = [1][2]\cdots[j]$ for $j \geq 1$, and set $[0]! = 1$. Thus $[1]! = 1$, $[2]! = 1+q$, $[3]! = (1+q)(1+q+q^2) = 1 + 2q + 2q^2 + q^3$, etc., and $[j]!_{q=1} = j!$. Finally define for $k \geq j \geq 0$,

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{[k]!}{[j]![k-j]!}.$$

The expression $\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right]$ is called a *q-binomial coefficient* (or *Gaussian coefficient*). Since $[r]!_{q=1} = r!$, it is clear that

$$\begin{bmatrix} k \\ j \end{bmatrix}_{q=1} = \binom{k}{j}.$$

One sometimes says that $\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right]$ is a "q-analogue of the binomial coefficient $\binom{k}{j}$."

6.4 Example. We have $\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right] = \left[\begin{smallmatrix} k \\ k-j \end{smallmatrix}\right]$ [why?]. Moreover,

$$\begin{bmatrix} k \\ 0 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = 1$$

$$\begin{bmatrix} k \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k-1 \end{bmatrix} = [k] = 1 + q + q^2 + \cdots + q^{k-1}$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{[4][3][2][1]}{[2][1][2][1]} = 1 + q + 2q^2 + q^3 + q^4$$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

In the above example, $\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right]$ was always a polynomial in q (and with nonnegative integer coefficients). It is not obvious that this is always the case, but it will follow easily from the following lemma.

6.5 Lemma. *We have*

$$\begin{bmatrix} k \\ j \end{bmatrix} = \begin{bmatrix} k-1 \\ j \end{bmatrix} + q^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}, \tag{26}$$

whenever $k \geq 1$, with the "initial conditions" $\left[{0 \atop 0}\right] = 1$, $\left[{k \atop j}\right] = 0$ if $j < 0$ or $j > k$ (the same intial conditions satisfied by the binomial coefficients $\binom{k}{j}$).

Proof. This is a straightforward computation. Specifically, we have

$$\begin{aligned}
\begin{bmatrix} k-1 \\ j \end{bmatrix} + q^{k-j}\begin{bmatrix} k-1 \\ j-1 \end{bmatrix} &= \frac{[k-1]!}{[j]![k-1-j]!} + q^{k-j}\frac{[k-1]!}{[j-1]![k-j]!} \\
&= \frac{[k-1]!}{[j-1]![k-1-j]!}\left(\frac{1}{[j]} + \frac{q^{k-j}}{[k-j]}\right) \\
&= \frac{[k-1]!}{[j-1]![k-1-j]!}\frac{[k-j]+q^{k-j}[j]}{[j][k-j]} \\
&= \frac{[k-1]!}{[j-1]![k-1-j]!}\frac{[k]}{[j][k-j]} \\
&= \begin{bmatrix} k \\ j \end{bmatrix}. \ \square
\end{aligned}$$

Note that if we put $q = 1$ in (26) we obtain the well-known formula

$$\binom{k}{j} = \binom{k-1}{j} + \binom{k-1}{j-1},$$

which is just the recurrence defining Pascal's triangle. Thus equation (26) may be regarded as a q-analogue of the Pascal triangle recurrence.

We can regard equation (26) as a recurrence relation for the q-binomial coefficients. Given the initial conditions of Lemma 6.5, we can use (26) inductively to compute $\left[{k \atop j}\right]$ for any k and j. From this it is obvious by induction that the q-binomial coefficient $\left[{k \atop j}\right]$ is a polynomial in q with nonnegative integer coefficients. The following theorem gives an even stronger result, namely, an explicit combinatorial interpretation of the coefficients.

6.6 Theorem. *Let $p_i(m,n)$ denote the number of elements of $L(m,n)$ of rank i. Then*

$$\sum_{i \geq 0} p_i(m,n)q^i = \begin{bmatrix} m+n \\ m \end{bmatrix}. \qquad (27)$$

(NOTE. The sum on the left-hand side is really a *finite* sum, since $p_i(m,n) = 0$ if $i > mn$.)

Proof. Let $P(m,n)$ denote the left-hand side of (27). We will show that

$$P(0,0) = 1, \text{ and } P(m,n) = 0 \text{ if } m < 0 \text{ or } n < 0 \tag{28}$$

$$P(m,n) = P(m,n-1) + q^n P(m-1,n). \tag{29}$$

Note that equations (28) and (29) completely determine $P(m,n)$. On the other hand, substituting $k = m+n$ and $j = m$ in (26) shows that $\left[{m+n \atop m}\right]$ also satisfies (29). Moreover, the initial conditions of Lemma 6.5 show that $\left[{m+n \atop m}\right]$ also satisfies (28). Hence (28) and (29) imply that $P(m,n) = \left[{m+n \atop m}\right]$, so to complete the proof we need only establish (28) and (29).

Equation (28) is clear, since $L(0,n)$ consists of a single point (the empty partition $\emptyset$), so $\sum_{i\geq 0} p_i(0,n)x^i = 1$; while $L(m,n)$ is empty (or undefined, if you prefer) if $m < 0$ or $n < 0$,

The crux of the proof is to show (29). Taking the coefficient of q^i of both sides of (29), we see [why?] that (29) is equivalent to

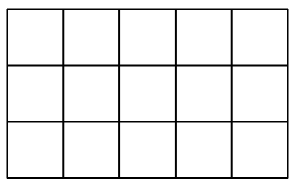
$$p_i(m,n) = p_i(m,n-1) + p_{i-n}(m-1,n). \tag{30}$$

Consider a partition $\lambda \vdash i$ whose Young diagram D fits in an $m\times n$ rectangle R. If D does not contain the upper right-hand corner of R, then D fits in an $m \times (n-1)$ rectangle, so there are $p_i(m,n-1)$ such partitions λ. If on the other hand D does contain the upper right-hand corner of R, then D contains the whole first row of R. When we remove the first row of R, we have left a Young diagram of size $i-n$ which fits in an $(m-1)\times n$ rectangle. Hence there are $p_{i-n}(m-1,n)$ such λ, and the proof follows [why?]. $\square$

Note that if we set $q = 1$ in (27), then the left-hand side becomes $|L(m,n)|$ and the right-hand side $\binom{m+n}{m}$, agreeing with Proposition 6.3.

NOTE: There is another well-known interpretation of $\left[{k \atop j}\right]$, this time not of its coefficients (regarded as a polynomial in q), but rather at its *values* for certain q. Namely, suppose q is the power of a prime. Recall that there is a field $\mathbb{F}_q$ (unique up to isomorphism) with q elements. Then one can show that $\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right]$ is equal to the number of j-dimensional subspaces of a k-dimensional vector space over the field $\mathbb{F}_q$. We will not discuss the proof here since it is not relevant for our purposes.

As the reader may have guessed by now, the poset $L(m,n)$ is isomorphic to a quotient poset B_s/G for a suitable integer $s > 0$ and finite group G acting on B_s. Actually, it is clear that we must have $s = mn$ since $L(m,n)$ has rank mn and in general B_s/G has rank s. What is not so clear is the right choice of G. To this end, let $R = R_{mn}$ denote an $m \times n$ rectangle of squares. For instance, R_{35} is given by the 15 squares of the diagram



We now define the group $G = G_{mn}$ as follows. It is a subgroup of the group $\mathfrak{S}_R$ of all permutations of the squares of R. A permutation π in G is allowed to permute the elements in each row of R in any way, and then to permute the rows themselves of R in any way. The elements of each row can be permuted in $n!$ ways, so since there are m rows there are a total of $n!^m$ permutations preserving the rows. Then the m rows can be permuted in $m!$ ways, so it follows that the order of G_{mn} is given by $m!n!^m$. (The group G_{mn} is called the *wreath product* of $\mathfrak{S}_n$ and $\mathfrak{S}_m$, denoted $\mathfrak{S}_n \wr \mathfrak{S}_m$ or $\mathfrak{S}_n$ wr $\mathfrak{S}_m$. However, we will not discuss the general theory of wreath products here.)

6.7 Example. Suppose $m = 4$ and $n = 5$, with the boxes of X labelled as follows.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20

Then a typical permutation π in $G(4,5)$ looks like

16	20	17	19	18
4	1	5	2	3
12	13	15	14	11
7	9	6	10	8

,

i.e., $\pi(16) = 1$, $\pi(20) = 2$, etc.

We have just defined a group G_{mn} of permutations of the set R_{mn} of squares of an $m \times n$ rectangle. Hence G_{mn} acts on the boolean algebra B_R of all subsets of the set R. The next lemma describes the orbits of this action.

6.8 Lemma. *Every orbit $\mathcal{O}$ of the action of G_{mn} on B_R contains exactly one Young diagram D (i.e., exactly one subset $D \subseteq R$ such that D is left-justified, and if λ_i is the number of elements of D in row i of R, then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$).*

Proof. Let S be a subset of R, and suppose that S has α_i elements in row i. If $\pi \in G_{mn}$ and $\pi \cdot S$ has β_i elements in row i, then $\beta_1, \ldots, \beta_m$ is just some permutation of $\alpha_1, \ldots, \alpha_m$ [why?]. There is a unique permutation $\lambda_1, \ldots, \lambda_m$ of $\alpha_1, \ldots, \alpha_m$ satisfying $\lambda_1 \geq \cdots \geq \lambda_m$, so the only possible Young diagram D in the orbit $\pi \cdot S$ is the one of shape $\lambda = (\lambda_1, \ldots, \lambda_m)$. It's easy to see that the Young diagram D_λ of shape λ is indeed in the orbit $\pi \cdot S$. For by permuting the elements in the rows of R we can left-justify the rows of S, and then by permuting the rows of R themselves we can arrange the row sizes of S to be in weakly decreasing order. Thus we obtain the Young diagram D_λ as claimed. $\square$

We are now ready for the main result of this section.

6.9 Theorem. *The quotient poset $B_{R_{mn}}/G_{mn}$ is isomorphic to $L(m,n)$.*

Proof. Each element of B_R/G_{mn} contains a unique Young diagram D_λ by Lemma 6.8. Moreover, two different orbits cannot contain the same Young diagram D since orbits are disjoint. Thus the map $\varphi : B_R/G_{mn} \to L(m,n)$

defined by $\varphi(\mathcal{O}_\lambda) = \lambda$ is a bijection (one-to-one and onto), where $\mathcal{O}_\lambda$ is the orbit containing D_λ. We claim that in fact φ is an isomorphism of partially ordered sets. We need to show the following: Let $\mathcal{O}$ and $\mathcal{O}^*$ be orbits of G_{mn} (i.e., elements of B_R/G_{mn}). Let D_λ and D_{λ^*} be the unique Young diagrams in $\mathcal{O}$ and $\mathcal{O}^*$, respectively. Then there exist $D \in \mathcal{O}$ and $D^* \in \mathcal{O}^*$ satisfying $D \subseteq D^*$ if and only if $\lambda \leq \lambda^*$ in $L(m,n)$.

The "if" part of the previous sentence is clear, for if $\lambda \leq \lambda^*$ then $D_\lambda \subseteq D_{\lambda^*}$. So assume there exist $D \in \mathcal{O}$ and $D^* \in \mathcal{O}^*$ satisfying $D \subseteq D^*$. The lengths of the rows of D, written in decreasing order, are $\lambda_1, \ldots, \lambda_m$, and similarly for D^*. Since each row of D is contained in a row of D^*, it follows that for each $1 \leq j \leq m$, D^* has at least j rows of size at least λ_j. Thus the length λ_j^* of the jth largest row of D^* is at least as large as λ_j. In other words, $\lambda_j \leq \lambda_j^*$, as was to be proved. $\square$

Combining the previous theorem with Theorem 5.9 yields:

6.10 Corollary. *The posets $L(m,n)$ are rank-symmetric, rank-unimodal, and Sperner.*

Note that the rank-symmetry and rank-unimodality of $L(m,n)$ can be rephrased as follows: The q-binomial coefficient $\left[\begin{smallmatrix} m+n \\ m \end{smallmatrix}\right]$ has symmetric and unimodal coefficients. While rank-symmetry is easy to prove (see Proposition 6.2), the unimodality of the coefficients of $\left[\begin{smallmatrix} m+n \\ m \end{smallmatrix}\right]$ is by no means apparent. It was first proved by J. Sylvester in 1878 by a proof similar to the one above, though stated in the language of the invariant theory of binary forms. For a long time it was an open problem to find a combinatorial proof that the coefficients of $\left[\begin{smallmatrix} m+n \\ m \end{smallmatrix}\right]$ are unimodal. Such a proof would give an explicit injection (one-to-one function) $\mu : L(m,n)_i \to L(m,n)_{i+1}$ for $i < \frac{1}{2}mn$. (One difficulty in finding such maps μ is to make use of the hypothesis that $i < \frac{1}{2}mn$.) Finally around 1989 such a proof was found by Kathy O'Hara. However, O'Hara's proof has the defect that the maps μ are not order-matchings. Thus her proof does not prove that $L(m,n)$ is Sperner, but only that it's rank-unimodal. It is an outstanding open problem in algebraic combinatorics to find an explicit order-matching $\mu : L(m,n)_i \to L(m,n)_{i+1}$ for $i < \frac{1}{2}mn$.

Note that the Sperner property of $L(m,n)$ (together with the fact that the largest level is in the middle) can be stated in the following simple terms: The largest possible collection $\mathcal{C}$ of Young diagrams fitting in an $m \times n$ rectangle such that no diagram in $\mathcal{C}$ is contained in another diagram in $\mathcal{C}$ is obtained by taking all the diagrams of size $\frac{1}{2}mn$. Although the statement of this fact requires almost no mathematics to understand, there is no known proof that doesn't use algebraic machinery. (The several known algebraic proofs are all closely related, and the one we have given is the simplest.) Corollary 6.10 is a good example of the efficacy of algebraic combinatorics.

An application to number theory. There is an interesting application of Corollary 6.10 to a number-theoretic problem. Fix a positive integer k. For a finite subset S of $\mathbb{R}^+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$, and for a real number $\alpha > 0$, define

$$f_k(S, \alpha) = \# \left\{ T \in \binom{S}{k} : \sum_{t \in T} t = \alpha \right\}$$

In other words, $f_k(S, \alpha)$ is the number of k-element subsets of S whose elements sum to α. For instance, $f_3(\{1, 3, 4, 6, 7\}, 11) = 2$, since $1 + 3 + 7 = 1 + 4 + 6 = 11$.

Given positive integers $k < n$, our object is to maximize $f_k(S, \alpha)$ subject to the condition that $\#S = n$. We are free to choose both S and α, but k and n are fixed. Call this maximum value $h_k(n)$. Thus

$$h_k(n) = \max_{\substack{\alpha \in \mathbb{R}^+ \\ S \subset \mathbb{R}^+ \\ \#S = n}} f_k(S, \alpha).$$

What sort of behavior can we expect of the maximizing set S? If the elements of S are "spread out," say $S = \{1, 2, 4, 8, \ldots, 2^{n-1}\}$, then all the subset sums of S are distinct. Hence for any $\alpha \in \mathbb{R}^+$ we have $f_k(S, \alpha) = 0$ or 1. Similarly, if the elements of S are "unrelated" (e.g., linearly independent over the rationals, such as $S = \{1, \sqrt{2}, \sqrt{3}, e, \pi\}$), then again all subset sums are distinct and $f_k(S, \alpha) = 0$ or 1. These considerations make it plausible that we should take $S = [n] = \{1, 2, \ldots, n\}$ and then choose α appropriately. In other words, we are led to the conjecture that for any $S \in \binom{\mathbb{R}^+}{n}$ and $\alpha \in \mathbb{R}^+$, we have

$$f_k(S, \alpha) \leq f_k([n], \beta), \tag{31}$$

for some $\beta \in \mathbb{R}^+$ to be determined.

First let us evaluate $f_k([n],\alpha)$ for any α. This will enable us to determine the value of β in (31). Let $S = \{i_1,\dots,i_k\} \subseteq [n]$ with

$$1 \le i_1 < i_2 < \cdots < i_k \le n, \quad i_1 + \cdots + i_k = \alpha. \tag{32}$$

Let $j_r = i_r - r$. Then (since $1+2+\cdots+k = \binom{k+1}{2}$)

$$n-k \ge j_k \ge j_{k-1} \ge \cdots \ge j_1 \ge 0, \quad j_1 + \cdots + j_k = \alpha - \binom{k+1}{2}. \tag{33}$$

Conversely, given $j_1,\dots,j_k$ satisfying (33) we can recover $i_1,\dots,i_k$ satisfying (32). Hence $f_k([n],\alpha)$ is equal to the number of sequences $j_1,\dots,j_k$ satisfying (33). Now let

$$\lambda(S) = (j_k, j_{k-1}, \dots, j_1).$$

Note that $\lambda(S)$ is a partition of the integer $\alpha - \binom{k+1}{2}$ with at most k parts and with largest part at most $n-k$. Thus

$$f_k([n],\alpha) = p_{\alpha-\binom{k+1}{2}}(k, n-k), \tag{34}$$

or equivalently,

$$\sum_{\alpha \ge \binom{k+1}{2}} f_k([n],\alpha) q^{\alpha-\binom{k+1}{2}} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

By the rank-unimodality (and rank-symmetry) of $L(n-k,k)$ (Corollary 6.10), the largest coefficient of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is the middle one, that is, the coefficient of $\lfloor k(n-k)/2 \rfloor$. It follows that for fixed k and n, $f_k([n],\alpha)$ is maximized for $\alpha = \lfloor k(n-k)/2 \rfloor + \binom{k+1}{2} = \lfloor k(n+1)/2 \rfloor$. Hence the following result is plausible.

6.11 Theorem. *Let $S \in \binom{\mathbb{R}^+}{n}$, $\alpha \in \mathbb{R}^+$, and $k \in \mathbb{P}$. Then*

$$f_k(S,\alpha) \le f_k([n], \lfloor k(n+1)/2 \rfloor).$$

Proof. Let $S = \{a_1,\dots,a_n\}$ with $0 < a_1 < \cdots < a_n$. Let T and U be distinct k-element subsets of S with the same element sums, say $T = \{a_{i_1},\dots,a_{i_k}\}$ and $U = \{a_{j_1},\dots,a_{j_k}\}$ with $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. Define $T^* = \{i_1,\dots,i_k\}$ and $U^* = \{j_1,\dots,j_k\}$, so $T^*, U^* \in \binom{[n]}{k}$. The crucial observation is the following:

Claim. The elements $\lambda(T^*)$ and $\lambda(U^*)$ are incomparable in $L(k, n-k)$, i.e., neither $\lambda(T^*) \leq \lambda(U^*)$ nor $\lambda(U^*) \leq \lambda(T^*)$.

Proof of claim. Suppose not, say $\lambda(T^*) \leq \lambda(U)^*$ to be definite. Thus by definition of $L(k, n-k)$ we have $i_r - r \leq j_r - r$ for $1 \leq r \leq k$. Hence $i_r \leq j_r$ for $1 \leq r \leq k$, so also $a_{i_r} \leq a_{j_r}$ (since $a_1 < \cdots < a_n$). But $a_{i_1} + \cdots + a_{i_k} = a_{j_1} + \cdots + a_{j_k}$ by assumption, so $a_{i_r} = a_{j_r}$ for all r. This contradicts the assumption that T and U are distinct and proves the claim.

It is now easy to complete the proof of Theorem 6.11. Suppose that $S_1, \ldots, S_r$ are distinct k-element subsets of S with the same element sums. By the claim, $\{\lambda(S_1^*), \ldots, \lambda(S_r^*)\}$ is an antichain in $L(k, n-k)$. Hence r cannot exceed the size of the largest antichain in $L(k, n-k)$. By Theorem 6.6 and Corollary 6.10, the size of the largest antichain in $L(k, n-k)$ is given by $p_{\lfloor k(n-k)/2 \rfloor}(k, n-k)$. By equation (34) this number is equal to $f_k([n], \lfloor k(n+1)/2 \rfloor)$. In other words,

$$r \leq f_k([n], \lfloor k(n+1)/2 \rfloor),$$

which is what we wanted to prove. $\square$

Note that an equivalent statement of Theorem 6.11 is that $h_k(n)$ is equal to the coefficient of $\lfloor k(n-k)/2 \rfloor$ in $\left[n \atop k \right]$ [why?].