Pak

11/21/05

$\alpha_n(G) = \sup_{\substack{H \leq G \\ |H| = n}} \alpha(H)$

$\alpha(G) = \limsup_{n \to \infty} \alpha_n(G)$

Thm (Marcus - Tardos)

$P = K \times K$ permutation matrix

$f(n, P) = \max \#$ 1's in $n \times n$ 0-1 matrix avoiding P

$f(n, P) \leq 2K^4 \binom{K^2}{K} n$

Cor $|\Sigma_n(\omega)| < c(\omega)^n$, where $c(\omega) = 15^{2K^4\binom{K^2}{K}}$

Pf:

Lemma: $f(n, P) \leq (K-1)^2 f(\frac{n}{K^2}, P) + 2K^3 \binom{K^2}{K} n$

Lemma $\Rightarrow$ thm By induction

$f(n, P) \leq (K-1)^2 f(\frac{n}{K^2}, P) + 2K^3 \binom{K^2}{K} n$

$\leq (K-1)^2 \, 2K^4 \binom{K^2}{K} n/K^2 + 2K^3 \binom{K^2}{K} n$

$= 2\binom{K^2}{K} n \left((K-1)^2 K^2 + K^3 \right) \leq 2K^4 \binom{K^2}{K} n$ ✓

(oKay, so you have to take ceilings + stuff, it's all trivial)

Pf of Lemma: Divvy up matrix A into $(n/K^2)^2$ block of size K^2, make new matrix B w/ 1's iff $\exists$ 1 in corresponding block. Let A_{ij} be the ij^{th} block.

Def'n A_{ij} is wide if $\exists$ 1's in at least K diff't columns, ~~[illegible]~~

Lemma: In a column of $(\frac{n}{K^2})$ blocks of A, at most $K\binom{K^2}{K}$ are wide Pf: Duh.

tall ≡ sim to wide

Now we're ready to prove the inequality

\# 1's in A ≤ ①\# 1s in wide blocks + ②\# 1s in tall blocks + ③ the rest

① $\leq K^4 \cdot \#$ wide blocks $\leq K^4 \cdot \frac{n}{K^2} K \binom{K^2}{K} = K^4 n \binom{K^2}{K}$

② $\leq$ "

③ $\leq \#$ 1's per block $\cdot$ # such blocks $\leq (K-1)^2 f(\frac{n}{K^2}, P)$ ✓

Conj ($100) (Arratia)

$c(w) \leq (K-1)^2$