18.306 Advanced Partial Differential Equations with Applications
Fall 2009

Lecture 23 2009 11 30 MON

TOPICS: Linear equations. Superposition. Normal modes and impulse problems (Green's functions). Heat equation in 1-D examples: various initial and boundary value problems. Method of images.

Linear equations:
Use superposition of special solutions to get general
Standard methods for evolution equations:
--- eigenmode analysis: separate time dependence as $e^{(\lambda t)}$
--- Greens functions: solve problems where the data is concentrated at a single location. Then integrate over these special solutions.
Decompose general problem into 3 special ones:
(1) Pure initial value. Homogeneous B.C. and no sources.
(2) Pure boundary value. Zero initial values and no sources.
(3) Pure sources. Zero initial values and homogeneous b.c.

EXAMPLES for the heat equation in 1-D. $T_t = T_{xx}$ % -----------------
#1 Initial values on the infinite line, no sources.% ----------------
-- Solution by normal modes. Fourier transform.
-- Green's Function. Use symmetries. Reduce problem to solving ode.
-- Connection between approaches. Fourier transform of a delta.
#2 Periodic initial values on the infinite line, no sources. % ------
-- Solution by normal modes. Fourier series.
-- Green's Function. Use periodic extension of Example #1 solution.
-- Note: Normal modes good for t large. Green's function expression we have good for short times.
#3 Initial data on half space $x > 0$. No sources. $T(0, t) = 0$.
-- Green's function by the method of images:
B.C. $T = 0$ equivalent to solution odd.
#4 Initial data on half space $x > 0$. No sources. $T_x(0, t) = 0$.
-- Green's function by the method of images:
B.C. $T_x = 0$ equivalent to solution even.
#5 Initial data on an interval with $T = 0$ at ends. No sources.
-- Green's function by the method of images, and periodic extension.

Other examples one can do: (a) $T = 0$ on one boundary and $T_x = 0$ on the other. (b) Robin boundary conditions. (c) Approach extends to simple sets in more than 1-D (later).

SYMMETRY: in all examples above $G(x, y, t) = G(y, x, t)$. Generic.
Motivate (no proof) by analogy with o.d.e. theory: $u_t = A*u$,
where A is symmetric. Then the solution operator $\exp(t*A)$,
is also symmetric.