18.305 Fall 2004/05

Assignment 7 Solutions: Boundary Layer Theory

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1. **Solve approximately**

$$\epsilon y'' + (1+x^2)y' + y = 0,\ 0 < x < 1, \epsilon << 1 \tag{1}$$

with boundary conditions

$$y(0) = y(1) = 1 \tag{2}$$

2. **Solve approximately**

$$\epsilon y'' + x(1+x)y' + \frac{1}{2}y = 0,\ 0 < x < 1, \epsilon << 1 \tag{3}$$

with boundary conditions

$$y(0) = 1 \text{ and } y(1) = 2 \tag{4}$$

3. **Solve approximately**

$$\epsilon y'' - 2xy' + (1+3x^3)y = 0,\ \ -1 < x < 1,\ \epsilon << 1 \tag{5}$$

with boundary conditions

$$y(-1) = 2 \text{ and } y(1) = 3 \tag{6}$$

Solutions:

1. Since $a(x) = -2\sin x \leq 0$, the rapidly varying solution is increasing with x. Thus there is a boundary layer of width ϵ near the endpoint $x = 1$. Also, since $a(x)$ has a simple zero at $x = 0$, there is a boundary layer of width $\sqrt{\epsilon}$ near the endpoint $x = 0$.

   We start by seeking the solution $y_{in}(x)$ valid inside the boundary layer near $x = 0$. This is because $y_r(x)$ is negligible in this region. Thus the solution in this region has only one arbitrary constant which can be determined from the boundary condition at $x = 0$.

   The solution inside the boundary layer near $x = 0$ is a linear combination of the solutions given by

$$y_{\pm} = e^{-\alpha x^2/4\epsilon} D_{\nu}(\pm\sqrt{\frac{|\alpha|}{\epsilon}}x) \tag{7}$$

   where

$$\nu = \frac{\beta}{|\alpha|} - \frac{sign(\alpha)+1}{2} \tag{8}$$

where $\beta = \cos(x)|_{x=0} = 1$, and $\alpha = a'(x)|_{x=0} = -2\cos 0 = -2$. Thus $\nu = 1/2$, and the solutions are

$$y_\pm = e^{-x^2/2\epsilon} D_{1/2}(\pm\sqrt{\frac{2}{\epsilon}}x) \tag{9}$$

The solution y_- is the rapidly increasing solution which is negligible inside the boundary layer near $x = 0$. Thus we have

$$y_{in}^{(near\ 0)}(x) = e^{x^2/2\epsilon} D_{1/2}(\sqrt{\frac{2}{\epsilon}}x)\frac{1}{D_{1/2}(0)} \tag{10}$$

where the boundary conditon $y(0) = 1$ has been utilized. Remembering that

$$D_\nu(X) \approx X^\nu e^{-X^2/4},\ X \to \infty \tag{11}$$

for $x >> \sqrt{\epsilon}$, we conclude

$$y_{in}^{(near\ 0)}(x) \approx \frac{1}{D_{1/2}(0)}(\sqrt{\frac{2}{\epsilon}}x)^{1/2} \tag{12}$$

Outside the boundary layers, we have

$$-2(\sin x)y' + (\cos x)y = 0 \tag{13}$$

which gives

$$y_{out}(x) = c\sqrt{\sin x} \tag{14}$$

Matching y_{out} with y_{in} in the region $1 >> x >> \sqrt{\epsilon}$, we obtain $c = \frac{1}{D_{1/2}(0)}(\frac{2}{\epsilon})^{1/4}$. Thus

$$y_{out}(x) = \frac{1}{D_{1/2}(0)}(\frac{2}{\epsilon})^{1/4}\sqrt{\sin x} \tag{15}$$

In particular, $y_{out}(1) = \frac{1}{D_{1/2}(0)}(\frac{2}{\epsilon})^{1/4}\sqrt{\sin 1}$.

Finally, we seek the the solution inside the boundary layer near $x = 1$. Since $a(1) = -2\sin 1$, we have

$$y_r(x) = [1 - y_{out}(1)]e^{-2\sin 1(1-x)/\epsilon} \tag{16}$$

Thus we have

$$y_{in}^{(near\ 1)}(x) = \frac{1}{D_{1/2}(0)}(\frac{2}{\epsilon})^{1/4}\sqrt{\sin 1} + [1 - \frac{1}{D_{1/2}(0)}(\frac{2}{\epsilon})^{1/4}\sqrt{\sin 1}]e^{-2\sin 1(1-x)/\epsilon} \tag{17}$$

2. Since $a(x) = x(1+x) \geq 0$, the rapidly varying solution is a decreasing function of x, hence there is a boundary layer near $x = 0$. Since $a(0) = 0$, which means that $x = 0$ is a turning point, the width of the boundary layer near $x = 0$ is of order $\sqrt{\epsilon}$.

   The rapidly varying solution y_r is negligible outside the boundary layer. Thus when $x >> \sqrt{\epsilon}$, the solution is approximately equal to y_{out} which satisfies

$$x(1+x)y'_{out} + \frac{1}{2}y_{out} = 0 \tag{18}$$

This equation yields $y_{out}(x) = c\sqrt{\frac{x+1}{x}}$ where c is a constant. Making use of the boundary condition at $x = 1$, we find

$$y_{out}(x) = \sqrt{\frac{2(x+1)}{x}} \tag{19}$$

Inside the boundary layer near $x = 0$, using the formulae given in the solution of problem 1,with $\alpha = 1, \beta = 1/2, \nu = -1/2$, we find

$$y_{in}(x) = e^{-x^2/4\epsilon}[c_1 D_{-1/2}(\sqrt{\frac{1}{\epsilon}}x) + c_2 D_{-1/2}(-\sqrt{\frac{1}{\epsilon}}x)] \tag{20}$$

where both c_1 and c_2 are arbitrary constants.We eliminate one of those constants by matching the solutions y_{in} and y_{out} in the region $1 >> x >> \sqrt{\epsilon}$,that is, by observing

$$y_{in}(x) \approx e^{-x^2/4\epsilon} c_2 D_{-1/2}(-\sqrt{\frac{1}{\epsilon}}x) \approx c_2 \frac{\sqrt{2\pi}}{\Gamma(1/2)}(\sqrt{\frac{1}{\epsilon}}x)^{-1/2} = c_2\sqrt{2}(\sqrt{\frac{1}{\epsilon}}x)^{-1/2} \tag{21}$$

by virtue of the formula

$$D_\nu(X) \approx \frac{\sqrt{2\pi}}{\Gamma(-\nu)}|X|^{-\nu-1}e^{X^2/4}, \ X \to \infty \tag{22}$$

This is because the first summand, $c_1 D_{-1/2}(-\sqrt{\frac{1}{\epsilon}}x)$, vanishes as $x >> \sqrt{\epsilon}$, which can be seen from (11).Also

$$y_{out}(x) \approx \sqrt{\frac{2}{x}} \tag{23}$$

in the region $1 >> x >> \sqrt{\epsilon}$. We note that both y_{in} and y_{out} are equal to a constant times $x^{-1/2}$ in the region $1 >> x >> \sqrt{\epsilon}$. This is an indication that this region is the overlapping region in which both approximations hold. Joining y_{in} and y_{out} in this overlapping region gives us

$$c_2 = (\frac{1}{\epsilon})^{1/4} \tag{24}$$

From the boundary condition at $x = 0$, we know $c_1 + c_2 = 1/D_{-1/2}(0)$, hence $c_1 = 1/D_{-1/2}(0) - (\frac{1}{\epsilon})^{1/4}$. Therefore

$$y_{in}(x) = e^{-x^2/4\epsilon}[(\frac{1}{D_{-1/2}(0)} - (\frac{1}{\epsilon})^{1/4})D_{-1/2}(\sqrt{\frac{1}{\epsilon}}x) + (\frac{1}{\epsilon})^{1/4}D_{-1/2}(-\sqrt{\frac{1}{\epsilon}}x)] \tag{25}$$

As a final observation, we note that $y_{out}(0)$ is infinite. But as we continue $y_{out}(x)$ into the region of the boundary layer, it turns into

$$(\frac{1}{\epsilon})^{1/4}e^{-x^2/4\epsilon}D_{-1/2}(-\sqrt{\frac{1}{\epsilon}}x) \tag{26}$$

At $x = 0$, the expression above is equal to $(\frac{1}{\epsilon})^{1/4}D_{-1/2}(0)$, which is a large number but not infinity.

3. We observe that, since $a(0) = 0$, there is turning point at $x = 0$, which is an interior point. Hence there is a boundary layer of width order $\sqrt{\epsilon}$ near $x = 0$. In this case, we know that the roles of the slowly varying solution and the rapidly varying solution interchange as one crosses the turning point $x = 0$.

   By the terminology of the notes and the book, we have $\alpha = -2$. Thus the negligible solution is the slowly varying solution, and the (possibly) order 1 solution is the rapidly varying solution.

   Also, the rapidly varying solution is increasing for $x > 0$. Therefore, there is a boundary layer of width ϵ near $x = 1$. Similarly, since the rapidly varying solution is decreasing for $x < 0$, there is a boundary layer of width of order ϵ near $x = -1$.(This is always the case if $\alpha < 0$, and there are no other turning points)

   Let us start with the slowly varying solution at $x = -1$. Since this solution becomes the rapidly varying solution in the region $x > 0$, and since the value of the solution at $x = 1$ is of order unity, this solution must be exponentially small outside the boundary layer near $x = 1$.

   Similarly if we start with the slowly varying solution at $x = 1$, and continue it to negative values of x, it becomes the rapidly varying solution in the region $x < 0$. Thus this solution must be exponentially small outside of the boundary layer at $x = -1$.

   The solution of the problem is the sum of the two solutions described above. It is appreciable only near the endpoints. Near $x = 1$, we have $a(x) = -2$, thus for $x > 0$

$$y_{in1}(x) \approx 3e^{-2(1-x)/\epsilon} \tag{27}$$

   Similarly, for $x < 0$

$$y_{in2}(x) \approx 2e^{-2(x+1)/\epsilon} \tag{28}$$

   where the boundary conditions are utilized. Finally, we can write

$$y_{uniform}(x) = 3e^{-2(1-x)/\epsilon} + 2e^{-2(x+1)/\epsilon} \tag{29}$$