MAKING MONEY FROM FAIR GAMES: EXAMINING THE BOREL-CANTELLI LEMMA

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Abstract. In this paper we discuss and prove the Borel-Cantelli Lemma. We then show two interesting applications of the Borel-Cantelli Lemma. These include a method by which one can profit from a series of games that individually have an expected value of zero, defying intuition regarding linearity of expectation, as well as displaying that a sequence of 100^{100} straight coin flips of heads will occur with probability one if one flips a fair coin into infinity.

1. Introduction

Consider an infinite sequence of games, where in game n one loses 2^n dollars with probability $\frac{1}{2^n+1}$ and win a dollar with probability $\frac{2^n}{2^n+1}$. Even though the expected value is 0, for all n, if we sum over n the probability that a person will lose in game n, we will find the total number of expected losses. For example, if we take the integral from 0 to infinity for $\frac{1}{2^n+1}\ dn$, we will find that

$$\begin{aligned}\int_0^\infty \frac{1}{2^n+1}dn &= n - \frac{\log(2^n+1)}{\log 2}\Big|_0^\infty \\ &= \infty - \frac{\log(2^\infty+1)}{\log 2} - 0 + \frac{\log(2^0+1)}{\log 2} \\ &= \infty - \infty - 0 + \frac{\log 2}{\log 2} \\ &= 1.\end{aligned}$$

We expect to lose a single time, even though we play into infinity and there always exists a non-zero probability of losing. We will explore this concept further by proving and utilizing the Borel-Cantelli Lemma, which states that if the sum of the individual probabilities for an infinite number of events is not infinite, then the number of events occurring as time goes to infinity is finite. For this paper, we mainly source the lecture notes of various university mathematics classes.

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2. Borel-Cantelli

The Borel-Cantelli Lemma states that if the sum of the probabilities of the events A_n is finite, then the set of all events that occur will also be finite. Note that no assumption of independence is required.

Conversely, the Borel-Cantelli Lemma can be used to show that if the sum of the probabilities of the independent events A_n is infinite, then the set of all events that will occur that are repeated infinitely many times must occur with probability one. That is, one will achieve an infinite number of these events as n moves to infinity. It is important to recognize that this converse Lemma requires an assumption of independence.

First, we will prove the finite case, where events only occur a finite number of times, even as n moves towards infinity. We use the notation $i.o.$ to represent "infinitely often."

It is also important to understand the concepts of the limit superior and the limit inferior. The limit inferior and limit superior of a sequence can be thought of as the limiting bounds of that sequence, typically as that sequence is taken to infinity. The limit inferior is the lower bound as the sequence trends to infinity, and the limit superior is the upper bound. For example, the limit superior of $f(x) = sin(x)$ as $x \to \infty$ is 1, while the limit inferior is -1. For our notation, we will refer to the limit superior as $lim\ sup$ and the limit inferior as $lim\ inf$.

We will prove this by showing that the probability of an event occurring infinite times as n trends to infinity is zero given the initial statement that the sum of the probabilities converges as n goes to infinity, meaning that the sum of the probabilities for discrete events in the sequence of events is less than infinity. If the probability of an event occurring ever goes to zero as n increases towards infinity, then we have shown that the number of times this event can happen is finite.

Lemma 2.1. *Let A_n be a sequence of events, where an individual event occurs at time n with probability $P(A_n)$. If the sum of the probabilities of A_n is finite, such that $\sum_{n=1}^{\infty} P(A_n) < \infty$, then the probability that infinitely many of these events occur is 0. That is, $P(\limsup_{n\to\infty} A_n) = 0$.*

Proof. Our goal is to prove that if

$$\sum_{n=1}^{\infty} P(A_n) < \infty,$$

then

$$P(A_n i.o.) = 0.$$

This states that if the sum of the probabilities is less than infinity for an infinite sequence of events, then the number of events is guaranteed to be finite. In essence, we are proving that the probability of an event happening infinitely often is zero.

$$\begin{aligned} P(A_n i.o.) &= P(\limsup_{n\to\infty} A_n) \\ &= \lim_{n\to\infty} P(\cup_{k=n}^{\infty} A_k) \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k). \end{aligned}$$

The final summation goes to 0 as n goes to infinity because we are working under the assumption that the series converges. Therefore, if the final term goes to zero, then

$$P(A_n i.o.) \leq 0.$$

and we have proven our Lemma, as the probability of the event happening infinitely often is zero. Probabilities cannot be negative.

□

For this lemma, the sequence of events does not have to be independent, a useful fact.

Now we will show the converse, namely that if the sum of the probabilites of the independent events A_n is infinite, then the set of all outcomes that are repeated infintely many times must occur with probability one. This Lemma differs from the previous in that it only holds for an independent sequence of events.

Lemma 2.2. *If the sum of the probabilities of an independent sequence of events A_n is infinite, such that $\sum_{n=1}^{\infty} Pr(A_n) > \infty$, then the probability that infinitely many of them occur is 1. That is, $Pr(\lim_{n\to\infty} sup\, A_n) = 1$.*

Proof. The Lemma states that if

$$\sum_{n=1}^{\infty} P(A_n) = \infty,$$

then the events will occur infinitely often as n goes to infinity, or

$$P(A_n i.o.) = 1.$$

We can use the complement of the probability that an event occurs infinitely often, or

$$1 - P(A_n i.o.),$$

to prove our Lemma, as well as the laws of independence, the limit inferior of the sequence, and the intersection of all complementary events. We are proving that the probability of an event happening only a finite number of times is zero. By proving this, we have proven our lemma that an event occurs an infinite number of times as n goes to infinity.

We recognize that

$$\begin{aligned} 1 - P(A_n i.o.) &= P(A_n i.o.)^c \\ &= P(\liminf_{n\to\infty} A_n^c) \\ &= \lim_{n\to\infty} P(\cap_{k=n}^{\infty} A_k^c). \end{aligned}$$

where the c superscript represents the complementary probability.

Because the events are independent with respect to each other, we can state that the complementary sequence of events are also independent. Therefore, because joint probabilities are simply the product of individual event probabilities, we see that for every n,

$$\begin{aligned} P(\cap_{k=n}^{\infty} A_k^c) &= \lim_{N\to\infty} P(\cap_{k=n}^{N} A_k^c) \\ &= \lim_{N\to\infty} \prod_{k=n}^{N} P(A_k^c) \\ &= \lim_{N\to\infty} \prod_{k=n}^{N} [1 - P(A_k)] \\ &\leq \lim_{N\to\infty} \prod_{k=n}^{N} \exp[-P(A_k)] \\ &= \lim_{N\to\infty} \exp[-\sum_{k=n}^{N} P(A_k)] \\ &= 0. \end{aligned}$$

We know that the penultimate term is equal to 0 because the series diverges, an assumption for this converse Lemma. The exponential function approaches 0 as the exponent approaches negative infinity, which the negative summation function does in the penultimate term.

Therefore, the probability of an event occurring a finite number of times is zero and we have proven the converse Lemma for an independent sequence of events. □

The result leads to an interesting corollary of these two Lemmas, sometimes known as the “0-1” law.

Corollary 2.3. *If $\{A_n\}_{n\geq 1}$ is a sequence of independent events in a probability space, then either $P(A(i.o.)) = 0$, the $\mathbb{E}(N) < \infty$ case, or $P(A(i.o.)) = 1$, the $\mathbb{E}(N) = \infty$ case, where N denotes the total number of A_n to occur:*

$$N = \sum_{n=1}^{\infty} I_n$$

where I_n denotes the indicator random variable for the event A_n.

It is easy to see the reasoning for the name, "0-1" law. Either the sum of the probabilities associated with the sequence of events is infinite, or it is not. Either one can expect the number of times an event happens to be infinite as time goes to infinity, or one can expect it to be finite. The outcome is binary; the probability of the event occurring into infinity is either 0 or 1 depending on whether the probability function converges or diverges towards infinity.

3. Application of the Borel-Cantelli Lemma

We can apply the Borel-Cantelli Lemma to an interesting situation where one can expect to profit from a series of games that, individually, have an expected value of 0. This should be counter-intuitive; because of linearity of expectation, the total expectancy of the series of games is still zero.

Take, for example, an infinite sequence of games, where in game n one loses 2^n dollars with probability $\frac{1}{2^n+1}$ and wins a dollar with probability $\frac{2^n}{2^n+1}$. This implies that one will lose a great deal of money a little bit of the time, and win a little bit of money most of the time. We can see that the expectation of any individual game is

$$\frac{-2^n}{2^n+1} + \frac{2^n}{2^n+1} = 0.$$

No matter which game in the series that one is playing, the expected value will always be zero. This is where the Borel-Cantelli lemma makes things interesting. We recognize that the expected number of losses is the sum over n of the probability that an individual will lose in game n. We showed in the introduction that

$$\int_{0}^{\infty} \frac{1}{2^n+1} dn = 1.$$

The function $\frac{1}{2^n+1}$ will be positive and decreasing for any n, $0 \leq n \leq \infty$. For this reason, the definite integral of the function between 0 and infinity will be greater than the sum for every integer n between 1 and infinity. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} \leq \int_0^{\infty} \frac{1}{2^n+1} dn \leq 1 < \infty$$

and accordingly is finite. A numerical approximation of

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

using java yields a sum of approximately 0.764, which, as we expected, is less than 1.

Because the sum of the probabilities is finite, it follows that with probability one the person will only lose a finite number of times, as shown by the Borel-Cantelli lemma. Therefore, the amount of money won by the player goes to infinity, as they win infinite times and lose only finite times! It is inconsequential that when the player loses, they lose a large amount, and that when they win they only win a small amount.

The key to this type of situation is that the probability of losing money converges as n goes to infinity. In this case, the probability summation of losing money converges, whereas if the denominator consisted of the term n, such as the probability of a loss occurring with $p = \frac{1}{n}$, then the probability summation does not converge as n goes to infinity, and the player will face a series of games from which they will not be able to profit.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

And by the Borel-Cantelli lemma, losses will also occur an infinite number of times as one plays into infinity. Do not be duped into playing a losing (or fair) game when the probability summation of losing does not converge as n goes to infinity.

4. Application of the Converse Borel-Cantelli Lemma

We can also use the converse Lemma for some nice applications. One such application involves the possibility of flipping 100^{100} heads in a row with a fair coin. We want to calculate $\mathbb{P}(E)$, where E is defined as the event that a run of 100^{100} heads, an extremely improbable event, occurs in an infinite sequence of independent coin tosses. Because E_n are independent events, we find that the probability of the sequence occurring in the nth block of 100^{100} coin tosses is

$$P(E_n = \frac{1}{(2^{100})^{100}} > 0) \Rightarrow \sum_{n=1}^{\infty} P(E_n) = \infty.$$

By the converse Borel-Cantelli Lemma, we recognize that if the sum of the probabilities of the event sequence is infinite, then

$$P(E) = 1.$$

Therefore, the probability that a run of 100^{100} heads occurs in an infinite sequence of independent coin tosses is 1.

5. Conclusion

We have proven the Borel-Cantelli Lemma, showing that if the sum of the probabilities of events is not infinite, as we sum over an infinite number of events, then the expected number of times that the event occurs is finite. We then proved the converse Lemma for an independent sequence of events. We have also displayed how these useful Lemmas can apply to interesting situations, such as one where we can find a strategy to profit from a series of zero expectation games, or recognize that enough monkeys typing away at enough typewriters will eventually produce Homer's Odyssey.

References

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