Practice Test 1

18.303 Linear Partial Differential Equations

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1 Given

You may assume the eigenvalues of the Sturm-Liouville problem

$$\begin{aligned} X'' + \lambda X &= 0, \qquad 0 < x < 1 \\ X(0) &= 0 \qquad X(1) = 0 \end{aligned}$$

are $\lambda_n = n^2\pi^2$ and $X_n(x) = \sin(nx)$, for $n = 1, 2, ...$, without derivation.

You may also assume the following orthogonality conditions for m, n positive integers:

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

2 Question

Consider the following heat problem in dimensionless variables

$$\begin{aligned} u_t &= u_{xx} + \frac{\pi^2}{4}u - b, \qquad 0 < x < 1, \qquad t > 0 \\ u(0,t) &= 0, \qquad u(1,t) = 0, \qquad t > 0 \\ u(x,0) &= u_0 \qquad 0 < x < 1. \end{aligned}$$

(a) [3 points] Explain in terms of a heated rod precisely what the problem models mathematically.

(b) [3 points] Derive the equilibrium solution

$$u_E(x) = \frac{4b}{\pi^2}\left[1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right]$$

It is insufficient to simply verify that the solution works.

(c) [3 points] Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$\begin{aligned} v_t &= v_{xx} + \frac{\pi^2}{4}v, \qquad 0 < x < 1, \qquad t > 0 \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 \\ v(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

where $f(x)$ will be determined by the transformation.

(d) [3 points] For an appropriate value of α show that the transformation $w(x,t) = e^{\alpha t}v(x,t)$ further simplifies the problem to

$$\begin{aligned} w_t &= w_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\ w(0,t) &= 0, \qquad w(1,t) = 0, \qquad t > 0 \\ w(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

(e) [8 points] Derive the solution

$$w(x,t) = \sum_{n=1}^{\infty} w_n(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{2\left(u_0 - 4b/\pi^2\right)}{2n-1} + \frac{32b(2n-1)}{\pi^2(4n-3)(4n-1)}\right)e^{-(2n-1)^2\pi^2 t}\sin((2n-1)\pi x)$$

and hence solve for $u(x,t) = u_E(x) + \sum_{n=1}^{\infty} u_n(x,t)$ using the earlier transformations.

(f) [4 points] Prove that the solution $u(x,t)$ is unique. [Hint: first show that $w(x,t)$ is unique].

(g) [6 points] Let $u_0 = 4b/\pi^2$. Show that

$$\left|\frac{u_2(x,t)}{u_1(x,t)}\right| \leq \frac{27}{35}e^{-8}, \qquad t \geq 1/\pi^2.$$

Hence show that

$$u(x,t) \approx u_E(x) + A_1 e^{-3\pi^2 t/4}\sin(\pi x)$$

is a good approximation for $t \geq 1/\pi^2$. Sketch $u = u_0$ and $u = u_E(x)$ for $0 < x < 1$ and comment on the physical significance of the sign of A_1.