Problems for 2D & 3D Heat and Wave Equations

18.303 Linear Partial Differential Equations

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1 Problem 1

A rectangular metal plate with sides of lengths L, H and insulated faces is heated to a uniform temperature of u_0 degrees Celsius and allowed to cool with three of its edges maintained at 0^o C and the other insulated. You may use dimensional coordinates, with PDE

$$u_t = \kappa \nabla^2 u, \qquad 0 \leq x \leq L, \qquad 0 \leq y \leq H.$$

The BCs are

$$u(0, y, t) = 0 = u(L, y, t), \qquad u(x, 0, t) = 0, \qquad \frac{\partial u}{\partial y}(x, H, t) = 0. \tag{1}$$

(i) Solve for $u(x, y, t)$ subject to an initial condition $u(x, y, 0) = 100$.

(ii) Find the smallest eigenvalue λ and the first term approximation (i.e. the term with $e^{-\lambda \kappa t}$).

(iii) For fixed $t = t_0 \gg 0$, sketch the level curves $u =$ constant as solid lines and the heat flow lines as dotted lines, in the xy-plane.

(iv) Of all rectangular plates of equal area, which will cool the slowest? Hint: for each type of plate, the smallest eigenvalue gives the rate of cooling.

(v) Does a square plate, side length L, subject to the BCs (1) cool more or less rapidly than a rod of length L, with insulated sides, and with ends maintained at 0^o C? You may use the results we derived in class for the rod, without derivation.

2 Problem 2

Haberman Problem 7.3.3, p. 287. Heat equation on a rectangle with different diffusivities in the x- and y-directions.

3 Problem 3

Haberman Problem 7.7.4 (a), p. 316. The pie-shaped membrane problem.

4 Problem 4

Find the eigenvalue λ and corresponding eigenfunction v for the 30^o-60^o-90^o right triangle (i.e. a right triangle that has these angles); v and λ satisfy

$$\begin{aligned} \nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D, \\ v &= 0 \quad \text{on} \quad \partial D, \end{aligned}$$

where $D = \left\{(x, y) : 0 < y < \sqrt{3}x, \quad 0 < x < 1\right\}$.

Hint: combine the eigenfunctions on the rectangle $D = \left\{(x, y) : 0 < x < 1, 0 < y < \sqrt{3}\right\}$ to obtain an eigenfunction on D that is positive on D. We know that the first eigenfunction can be characterized (up to a non-zero multiplicative constant) as the eigenfunction that is of one sign. You may use the eigenfunctions derived in-class for the rectangle, without derivation. Be sure to sketch the region correctly before solving the problem.

5 Problem 5

Consider the boundary value problem on the isosceles right angled triangle of side length 1,

$$\nabla^2 v = 0, \qquad 0 < y < x, \qquad 0 < x < 1$$

subject to the BCs

$$\begin{aligned} \frac{\partial v}{\partial x}(1, y) &= 0, \qquad 0 < y < 1 \\ \frac{\partial v}{\partial y}(x, 0) &= 0, \qquad 0 < x < 1 \\ v(x, x) &= 0, \qquad 0 < x < 1/2 \\ v(x, x) &= 50, \qquad 1/2 < x < 1 \end{aligned}$$

Give a symmetry argument to show that $v(x, 1-x) = 25$ for $0 < x < 1$. Sketch the level curves and heat flow lines of v. Be sure to sketch the region correctly before solving the problem.