A few important PDEs

	constant coefficients = 1	(one of several possible examples of) *variable coefficients* = $c(\mathbf{x})$
Poisson's equation:	$\nabla^2 u = f$	$\nabla \cdot (c \nabla u) = f$
	example: f = charge density, u = –electric potential	c = permittivity ε
	example: f = heat source/sink rate u = steady-state temperature	c = thermal conductivity
	example: f = solute source/sink rate, u = steady-state concentration	c = diffusion coefficient
	example: $f \sim$ force on stretched string/drum u = steady-state displacement	$c \sim$ "springy-ness"
Laplace's equation:	$\nabla^2 u = 0$	$\nabla \cdot (c \nabla u) = 0$
	examples: as for Poisson, but no sources	
Heat/diffusion equation:	$\frac{\partial u}{\partial t} = \nabla^2 u$	$\frac{\partial u}{\partial t} = \nabla \cdot (c \nabla u)$
	examples: u = temperature	c = thermal conductivity
	u = solute concentration	c = diffusion coefficient
Scalar wave equation:	$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$	$\frac{\partial^2 u}{\partial t^2} = \nabla \cdot (c \nabla u)$
	examples: u = displacement of stretched string/drum u = density of gas/fluid	$c^2 = 1$ / wave speed

+ many, many others…

Maxwell (electromagnetism)

Schrödinger (quantum mechanics)

Navier–Stokes / Stokes / Euler (fluids)

Black-Scholes (options pricing)

Lamé–Navier (linear elastic solids)

beam equation (bending thin solid strips)

advection-diffusion (diffusion in flows)

reaction-diffusion (diffusion+chemistry)

minimal-surface equation (soap films)

nonlinear wave equation (e.g. solitary ocean waves)

18.06 / 18.303

	18.06 *finite-dimensional linear algebra*		18.303 *linear algebra w/ functions & derivatives*
unknowns:	vector space of *column vectors* $\mathbf{x}$ (or $\vec{x}$) in $\mathbb{R}^n$ (or $\mathbb{C}^n$), or possibly $\mathbf{x}(t)$ [time-dependent]	**vector space**: *we can add, subtract, & multiply by constants without leaving the space*	vector space of real-valued (or complex) *functions* $u(\mathbf{x})$ [for $\mathbf{x}$ in some *domain* Ω], or possibly $u(\mathbf{x},t)$ [time-dependent], … possibly restricted by some *boundary conditions* at the boundary $\partial\Omega$ [e.g. $u(\mathbf{x}) = 0$ on $\partial\Omega$] … possibly with vector-valued $\mathbf{u}(\mathbf{x})$ [vector fields]
linear operators:	*matrices* A	**linearity**: $A(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$ $\hat{A}(\alpha u+\beta v) = \alpha\hat{A}u + \beta\hat{A}v$	linear *operators on functions* $\hat{A}$, [$\hat{A}u$ = *function*] using *partial derivatives*. **examples**: $\hat{A}_1 u = \nabla^2 u$ [*Laplacian* operator] $\hat{A}_2 u = 3u$ [mult. by constant] $\hat{A}_3 u \mid_{\mathbf{x}} = a(\mathbf{x})\, u(\mathbf{x})$ [mult. by function] $\hat{A} = 4\hat{A}_1 + \hat{A}_2 + 7\hat{A}_3$ [linear comb. of ops.]
dot product and transpose:	$\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^*\mathbf{y} = \Sigma_i\, \bar{x}_i y_i$ $\mathbf{x}\cdot A\mathbf{y} = \mathbf{x}^*A\mathbf{y} = (A\mathbf{x})^*\mathbf{y}$ $\Leftrightarrow (A)^*_{ij} = \bar{A}_{ji}$ [conjugate & swap rows/cols]	complex $\mathbf{x}$: $\mathbf{x}^T \to \bar{\mathbf{x}}^T = \mathbf{x}^*$ $\left(\frac{\partial}{\partial x}\right)^* = ???$	$u(\mathbf{x})\cdot v(\mathbf{x}) = \langle u,v\rangle$ = ???????? [*inner product*] [= some integral] $\langle u,\hat{A}v\rangle = \langle\hat{A}^*u,v\rangle$ $\Rightarrow \hat{A}^*$ = ???????? (= $\hat{A}^\dagger$ in physics) [*adjoint*]
basis:	set of vectors $\mathbf{b}_i$ with span = whole space $\Leftrightarrow$ any $\mathbf{x} = \Sigma_i\, c_i\, \mathbf{b}_i$ for some coefficients c_i … if *orthonormal basis*, then $c_i = \mathbf{b}_i^*\mathbf{x}$	[e.g. Fourier series!]	∞ set of functions $b_i(\mathbf{x})$ with span = whole space $\Leftrightarrow$ any $u(\mathbf{x}) = \Sigma_i\, c_i\, b_i(\mathbf{x})$ for some coefficients c_i … if *orthonormal basis*, then $c_i = \langle b_i, u\rangle$
linear equations:	solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$		solve $\hat{A}u = f$ for $u(\mathbf{x})$
existence & uniqueness:	$A\mathbf{x} = \mathbf{b}$ solvable if $\mathbf{b}$ in column space of A. Solution unique if null space of $A = \{\mathbf{0}\}$, or equivalently if eigenvalues of A are $\neq 0$.		$\hat{A}u = f$ solvable if $f(\mathbf{x})$ in col. space (*image*) of $\hat{A}$. Solution unique if null space (*kernel*) of $\hat{A} = \{0\}$, or equivalently if eigenvalues of $\hat{A}$ are $\neq 0$.
eigenvalues/vectors:	solve $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ and λ. For this $\mathbf{x}$, A acts just like a number (λ). [e.g. $A^n\mathbf{x} = \lambda^n\mathbf{x}$, $e^A\mathbf{x} = e^\lambda\mathbf{x}$.]		solve $\hat{A}u = \lambda u$ for $u(\mathbf{x})$ [*eigenfunction*] and λ. For this u, $\hat{A}$ acts just like a number (λ). [e.g. $\hat{A}^n u = \lambda^n u$, $e^{\hat{A}}u = e^\lambda u$.] example: $\frac{\partial^2}{\partial x^2}\sin(kx) = \left(-k^2\right)\sin(kx)$
time-evolution initial-value problem:	solve $d\mathbf{x}/dt = A\mathbf{x}$ for $\mathbf{x}(0)=\mathbf{b}$ [system of *ODE*s] $\Rightarrow \mathbf{x} = e^{At}\,\mathbf{b}$ [if A constant] … expand $\mathbf{b}$ in eigenvectors, mult. each by $e^{\lambda t}$		solve $\partial u/\partial t = \hat{A}u$ for $u(\mathbf{x},0)=f(\mathbf{x})$ $\Rightarrow u(\mathbf{x},t) = e^{\hat{A}t}f(\mathbf{x})$ [if $\hat{A}$ constant] … expand f in eigenfunctions, mult. each by $e^{\lambda t}$
real-symmetric or Hermitian:	$A = A^*$ $\Rightarrow$ real λ, orthogonal eigenvectors, diagonalizable		$\hat{A} = \hat{A}^*$ [??????] $\Rightarrow$ real λ, orthogonal eigenvectors (???) diagonalizable (???)
positive definite / semi-definite:	$A = A^*$, $\mathbf{x}^*A\mathbf{x} > 0$ for any $\mathbf{x} \neq \mathbf{0}$ / $\mathbf{x}^*A\mathbf{x} \geq 0$ $\Leftrightarrow$ real $\lambda>0/\geq 0$, $A=B^*B$ for some B	important fact: $-\nabla^2$ is symmetric positive definite or semi-definite!	$\hat{A} = \hat{A}^*$, $\langle u,\hat{A}u\rangle>0$ / ≥ 0 for $u \neq 0$ (???) $\Leftrightarrow$ real $\lambda>0/\geq 0$, $\hat{A}=\hat{B}^*\hat{B}$ for some $\hat{B}$ (???)
inverses:	$A^{-1}A = AA^{-1} = 1$ [if it exists] $\Rightarrow A\mathbf{x}=\mathbf{b}$ solved by $\mathbf{x} = A^{-1}\mathbf{b}$	$\left(\frac{\partial}{\partial x}\right)^{-1} = ???$ … some kind of integral?	$\hat{A}^{-1}$ = ?????? $\Rightarrow \hat{A}u = f$ solved by $f = \hat{A}^{-1}u$??? […delta functions & Green's functions]
(real) orthogonal or unitary:	$A^{-1} = A^* \Leftrightarrow (A\mathbf{x})\cdot(A\mathbf{x}) = \mathbf{x}\cdot\mathbf{x}$ for any $\mathbf{x}$ $\Rightarrow \lvert\lambda\rvert=1$, orthogonal eigenvectors, diagonalizable		$\hat{A}^{-1} = \hat{A}^* \Leftrightarrow \langle\hat{A}u,\hat{A}u\rangle = \langle u,u\rangle$ for any u $\Rightarrow \lvert\lambda\rvert=1$, orthogonal eigenvectors (???) diagonalizable (???)