

## 2. THE STEEPEST DESCENT AND STATIONARY PHASE FORMULAS

Now, let us forget for a moment that the integrals (1,2,3) are infinite dimensional and hence problematic to define, and ask ourselves the following question: why should we expect that when the parameter  $\kappa$  or  $\hbar$  goes to zero, we recover the usual classical mechanics or field theory? The answer is that this expectation is based on the *steepest descent* (respectively, *stationary phase*) principle from classical analysis: if  $f(x)$  is a function in  $\mathbb{R}^d$  then the integrals  $\int g(x)e^{-f(x)/\kappa}dx$ ,  $\int g(x)e^{if(x)/\hbar}dx$  “localize” to minima, respectively critical points, of the function  $f$ . As this classical fact is of central importance to the whole course, let us now discuss it in some detail.

**2.1. The steepest descent formula.** Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be smooth functions.

**Theorem 2.1.** (*The steepest descent formula*) Assume that  $f$  attains a global minimum at a unique point  $c \in (a, b)$ , such that  $f''(c) > 0$ . Then one has

$$(4) \quad \int_a^b g(x)e^{-f(x)/\hbar}dx = \hbar^{1/2}e^{-f(c)/\hbar}I(\hbar),$$

where  $I(\hbar)$  extends to a smooth function on  $[0, \infty)$  such that  $I(0) = \sqrt{2\pi} \frac{g(c)}{\sqrt{f''(c)}}$ .

*Proof.* Let  $I(\hbar)$  be defined by the equation (4).

Let  $\epsilon$  be a real number, such that  $\frac{1}{2} > \epsilon > 0$ , and let  $I_1(\hbar)$  be defined by the same equation, but with integration over  $[c - \hbar^{\frac{1}{2}-\epsilon}, c + \hbar^{\frac{1}{2}-\epsilon}]$ . It is clear that  $I(\hbar) - I_1(\hbar)$  is “rapidly decaying in  $\hbar$ ” (i.e. it is  $O(\hbar^N)$ ,  $\hbar \rightarrow 0$  for any  $N$ ). So it suffices to prove the theorem for  $I_1(\hbar)$ .

Further, let us make in the integral defining  $I_1(\hbar)$  the change of variables  $y = (x - c)/\sqrt{\hbar}$ . Then we get

$$(5) \quad I_1(\hbar) = \int_{-\hbar^{-\epsilon}}^{\hbar^{-\epsilon}} g(c + y\sqrt{\hbar})e^{(f(c) - f(c + y\sqrt{\hbar}))/\hbar}dy.$$

Now, note that the integrand is a smooth function with respect to  $\sqrt{\hbar}$  for  $\hbar \geq 0$ . Let  $I_2(\hbar)$  be the same integral as in (5) but with integrand replaced by its Taylor expansion in  $\sqrt{\hbar}$  at 0 modulo  $\hbar^N$ . Then  $|I_1(\hbar) - I_2(\hbar)| \leq C\hbar^{N-\epsilon}$ .

Finally, let  $I_3(\hbar)$  be defined by the same integral as  $I_2(\hbar)$  but with limits from  $-\infty$  to  $\infty$ . Then  $I_2(\hbar) - I_3(\hbar)$  is rapidly decaying in  $\hbar$ .

Thus, it suffices to show that  $I_3(\hbar)$  admits a Taylor expansion in  $\hbar^{1/2}$  modulo  $\hbar^{N-\epsilon}$ , and that the value at zero is as stated. But we know that  $I_3(\hbar)$  is a polynomial in  $\sqrt{\hbar}$ . Also, the integrals giving coefficients of non-integer powers of  $\hbar$  are integrals over  $\mathbb{R}$  of odd functions, so they are zero. So the first statement (existence of the Taylor expansion) is proved. The value  $I_3(0)$  is given by the integral

$$g(c) \int_{-\infty}^{\infty} e^{-\frac{f''(c)y^2}{2}} dy,$$

so it is computed from the well known Poisson integral:

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi}.$$

The theorem is proved. □

**2.2. Stationary phase formula.** This theorem has the following imaginary analog, called the stationary phase formula.

**Theorem 2.2.** Assume that  $f$  has a unique critical point  $c \in (a, b)$ , with  $f''(c) \neq 0$ , and  $g$  vanishes with all derivatives at  $a, b$ . Then

$$\int_a^b g(x)e^{if(x)/\hbar}dx = \hbar^{1/2}e^{if(c)/\hbar}I(\hbar),$$

where  $I(\hbar)$  extends to a smooth function on  $[0, \infty)$  such that  $I(0) = \sqrt{2\pi}e^{\pm\pi i/4} \frac{g(c)}{|\sqrt{f''(c)}|}$ , where  $\pm$  is the sign of  $f''(c)$ .

**Remark.** It is important to assume that  $g$  vanishes with all derivatives at the ends of the integration interval. Otherwise we will get some additional boundary contributions.

*Proof.* (sketch). The proof is analogous to the real case, but slightly more subtle. The differences are as follows. First of all, the Poisson integral is replaced with the (conditionally convergent) Fresnel integral

$$\int_{-\infty}^{\infty} e^{\frac{iy^2}{2}} dy = \sqrt{2\pi} e^{\pi i/4}.$$

Further, one should partition  $g$  in a sum of two smooth functions, one localized around  $c$  on an interval of size  $2\hbar^{1/2-\epsilon}$ , and the other vanishing near  $c$ . Next, one needs to show that only the first summand matters, by using Riemann's lemma: if  $f$  has no critical points in the support of  $g$  then  $\int_a^b g(x)e^{if(x)/\hbar}$  is rapidly decaying (prove this!). Finally, for  $g$  localized around  $c$ , one makes the change of variable like in the real case. The statement about existence of Taylor expansion is proved as in the real case, and the value at 0 is calculated using Fresnel integral.  $\square$

**2.3. Non-analyticity of  $I(\hbar)$  and Borel summation.** It is very important to note that the Taylor series for  $I(\hbar)$  is usually not convergent and is only an asymptotic expansion, so that the function  $I$  is smooth but not analytic at zero. To illustrate this, consider the integral

$$\int_{-\infty}^{\infty} e^{-\frac{x^2+x^4}{2\hbar}} dx = \sqrt{2\pi\hbar}^{1/2} I(\hbar),$$

where

$$I(\hbar) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2+\hbar y^4}{2}} dy.$$

The latter integral expands asymptotically as

$$I(\hbar) = \sum_{n=0}^{\infty} a_n \hbar^n,$$

where

$$a_n = \frac{(-1)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \frac{y^{4n}}{2^n n!} dy = \frac{(-1)^n}{\sqrt{2\pi}} 2^{n+\frac{1}{2}} \Gamma(2n + \frac{1}{2}) / n!.$$

It is clear that this sequence has superexponential growth, so the radius of convergence of the series is zero.

**Remark.** In fact, the non-analyticity of  $I(\hbar)$  is related to the fact that the integral defining  $I(\hbar)$  is divergent for  $\hbar < 0$ .

Let us now discuss the question: to what extent does the asymptotic expansion of the function  $I(\hbar)$  (which we can find using Feynman diagrams as explained below) actually determines this function? Suppose that  $\tilde{I}(\hbar) = \sum_{n \geq 0} a_n \hbar^n$  is a series with zero radius of convergence. In general, we cannot uniquely determine a function  $I$  on  $[0, \epsilon)$  whose expansion is given by such a series. However, assume that  $a_i$  are such that the series  $g(\hbar) = \sum_{n \geq 0} a_n \hbar^n / n!$  is convergent in some neighborhood of 0, analytically continues to  $[0, \infty)$ , and has at most exponential growth as  $\hbar \rightarrow \infty$ . In this case there is a "canonical" way to construct a smooth function  $I$  on  $[0, \epsilon)$  with (asymptotic) Taylor expansion  $\tilde{I}$ , called *Borel summation* of  $\tilde{I}$ . Namely, the function  $I$  is defined by the formula

$$I(\hbar) = \int_0^{\infty} g(\hbar u) e^{-u} du$$

The fact that  $I$  has the Taylor expansion  $\tilde{I}$  follows from the fact that for  $t > 0$  one has

$$\int_0^{\infty} x^n e^{-x} dx = n!.$$

For example, consider the series  $\tilde{I} = \sum_{n \geq 0} (-1)^n n! \hbar^n$ . Then  $g(\hbar) = \sum_{n \geq 0} (-1)^n \hbar^n = \frac{1}{1+\hbar}$ . Hence, the Borel summation yields  $I(\hbar) = \int_0^{\infty} \frac{e^{-u}}{1+\hbar u} du$ .

Physicists expect that in many situations perturbation expansions in quantum field theory are Borel summable, and the actual answers are obtained from these expansions by Borel summation. The Borel summability of perturbation series has actually been established in a few nontrivial examples of QFT.

**2.4. Application of steepest descent.** Let us give an application of Theorem 2.1. Consider the integral

$$\int_0^\infty t^s e^{-t} dt, s > 0.$$

It is well known that this integral is equal to the Gamma function  $\Gamma(s + 1)$ . By doing a change of variable  $t = sx$ , we get

$$\frac{\Gamma(s + 1)}{s^{s+1}} = \int_0^\infty x^s e^{-sx} dx = \int_0^\infty e^{-s(x - \log x)} dx.$$

Thus, we can apply Theorem 2.1 for  $\hbar = 1/s$ ,  $f(x) = x - \log x$ ,  $g(x) = 1$  (of course, the interval  $[a, b]$  is now infinite, and the function  $f$  blows up on the boundary, but one can easily see that the theorem is still applicable). The function  $f(x) = x - \log x$  has a unique critical point on  $[0, \infty)$ , which is  $c = 1$ , and we have  $f''(c) = 1$ . Then we get

$$\Gamma(s + 1) = s^s e^{-s} \sqrt{2\pi s} (1 + a_1/s + a_2/s^2 + \dots).$$

This is the celebrated Stirling's formula.

**2.5. Multidimensional versions of steepest descent and stationary phase.** Theorems 2.1, 2.2 have multidimensional analogs. To formulate them, let  $V$  be a real vector space of dimension  $d$  with a fixed volume element  $dx$ , and let  $f, g$  be smooth functions in a closed box  $\mathbf{B} \subset V$ .

**Theorem 2.3.** *Assume that  $f$  has global minimum on  $\mathbf{B}$  at a unique interior point  $c$ , such that  $f''(c) > 0$ . Then*

$$(6) \quad \int_{\mathbf{B}} g(x) e^{-f(x)/\hbar} dx = \hbar^{d/2} e^{-f(c)/\hbar} I(\hbar),$$

where  $I(\hbar)$  extends to a smooth function on  $[0, \infty)$  such that  $I(0) = (2\pi)^{d/2} \frac{g(c)}{\sqrt{\det f''(c)}}$ .

**Theorem 2.4.** *Assume that  $f$  has a unique critical point  $c$  in  $\mathbf{B}$ , such that  $\det f''(c) \neq 0$ , and that  $g$  vanishes with all derivatives on the boundary of the box. Then*

$$(7) \quad \int_{\mathbf{B}} g(x) e^{if(x)/\hbar} dx = \hbar^{d/2} e^{if(c)/\hbar} I(\hbar),$$

where  $I(\hbar)$  extends to a smooth function on  $[0, \infty)$  such that  $I(0) = (2\pi)^{d/2} e^{\pi i \sigma / 4} \frac{g(c)}{\sqrt{|\det f''(c)|}}$ , where  $\sigma$  is the signature of the symmetric bilinear form  $f''(c)$ .

**Remark.** In presence of a volume element on  $V$ , the determinant of a symmetric bilinear form is well defined.

The proofs of these theorems are parallel to the proofs of their one dimensional versions. Namely, the 1-dimensional Poisson and Fresnel integrals are replaced with their multidimensional versions – the Gaussian integrals

$$\int_V e^{-B(x,x)/2} dx = (2\pi)^{d/2} (\det B)^{-1/2},$$

for a symmetric bilinear form  $B > 0$ , and

$$\int_V e^{iB(x,x)/2} dx = (2\pi)^{d/2} e^{\pi i \sigma(B)/4} |\det B|^{-1/2},$$

for nondegenerate- $B$ . These integral formulas are easily deduced from the one-dimensional ones by diagonalizing the bilinear form- $B$ .