Graph Theory and Additive Combinatorics

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8
The sum-product problem

In this chapter, we consider how sets behave under both addition and multiplication. The main problem, called ***the sum-product problem***, is the following: can $A + A$ and $A \cdot A = \{ab : a, b \in A\}$ both be small for the same set A?

We take an example $A = [N]$. Then $|A + A| = 2N - 1$, but it turns out that the product set has a large size, $|A \cdot A| = N^{2-o(1)}$. The problem of determining the size of the product set is known as *Erdős multiplication table problem*. One can also see that if A is a geometric progression, then $A \cdot A$ is small, yet $A + A$ is large. The main conjecture concerning the sum-product problem says that either the sum set or the product set has the size very close to the maximum.

Ford (2008)

Conjecture 8.1 (Erdős–Szemerédi's conjecture). *For every finite subset A of $\mathbb{R}$, we have*

Erdős and Szemerédi (1983)

$$\max\{|A + A|, |A \cdot A|\} \geq |A|^{2-o(1)}$$

In this chapter, we will see two proofs of lower bounds on the sum-product problem. To do this, we first develop some tools.

8.1 *Crossing number inequality*

The ***crossing number*** $\mathrm{cr}(G)$ of a graph G is defined to be the minimum number of crossings in a planar drawing of G with curves. Given a graph with many edges, how big must its crossing number be?

Theorem 8.2 (Crossing number inequality). *If $G = (V, E)$ is a graph satisfying $|E| \geq 4|V|$, then $cr(G) \geq c|E|^3/|V|^2$ for some constant $c > 0$.*

Ajtai, Chvátal, Newborn and Szemerédi (1982)
Leighton (1984)

It follows directly that every n-vertex graph with $\Omega(n^2)$ edges has $\Omega(n^4)$ crossings.

Proof of Theorem 8.2. For any connected planar graph with at least one cycle, we have $3|F| \leq 2|E|$, with $|F|$ denoting the number of faces. The inequality follows from double-counting of faces using that every face is adjacent to at least three edges and that every edge is adjacent to at most two faces. Applying Euler's formula, we get $|E| \leq 3|V| - 6$. Therefore $|E| \leq 3|V|$ holds for every planar graph G including ones that are not connected or do not have a cycle. Thus we have $cr(G) > 0$ if $|E| > 3|V|$.

Let G be a finite, connected, planar graph and suppose that G is drawn in the plane without any edge intersection. Euler's formula states $|V| - |E| + |F| = 2$.

Suppose G satisfies $|E| > 3|V|$. Since we can get a planar graph by deleting each edge that witnesses a crossing, we have $|E| - \mathrm{cr}(G) \geq 3|V|$. Therefore

$$\mathrm{cr}(G) \geq |E| - 3|V|. \tag{8.1}$$

In order to get the desired inequality, we use a trick from the probabilistic method. Let $p \in [0, 1]$ be some real number to be determined and let $G' = (V', E')$ be a graph obtained by randomly keeping each vertex of G with probability p iid. By (8.1), we have $\mathrm{cr}(G') \geq |E'| - 3|V'|$ for every G'. Therefore the same inequality must hold if we take the expected values of both sides:

$$\mathbb{E}\,\mathrm{cr}(G') \geq \mathbb{E}|E'| - 3\mathbb{E}|V'|.$$

One can see that $\mathbb{E}|E'| = p^2|E|$ since an edge remains if and only if both of its endpoints are kept. Similarly $\mathbb{E}|V'| = p|V|$. By keeping the same drawing, we get the inequality $p^4\,\mathrm{cr}(G) \geq \mathbb{E}\,\mathrm{cr}(G')$. Therefore we have

$$\mathrm{cr}(G) \geq p^{-2}|E| - 3p^{-3}|V|.$$

Finally we get the desired inequality by setting $p \in [0, 1]$ so that $4p^{-3}|V| = p^{-2}|E|$, which can be done from the condition $|E| \geq 4|V|$. $\square$

8.2 *Incidence geometry*

Another field in mathematics related to the sum-product problem is incidence geometry. The ***incidence*** between the set of points $\mathcal{P}$ and the set of lines $\mathcal{L}$ is defined as

$$I(\mathcal{P}, \mathcal{L}) = |\{(p, \ell) \in \mathcal{P} \times \mathcal{L} : p \in \ell\}|$$

What's the maximum number of incidences between n points and n lines? One trivial upper bound is $|\mathcal{P}||\mathcal{L}|$. We can get a better bound by using the fact that every pair of points is determined by at most one line:

$$\begin{aligned}
|\mathcal{P}|^2 &\geq \#\{(p, p', \ell) \in \mathcal{P} \times \mathcal{P} \times \mathcal{L} : pp' \in \ell\,,\, p \neq p'\} \\
&\geq \sum_{\ell \in \mathcal{L}} |\mathcal{P} \cup \ell|(|\mathcal{P} \cup \ell| - 1) \\
&\geq \frac{I(\mathcal{P}, \mathcal{L})^2}{|\mathcal{L}|^2} - I(\mathcal{P}, \mathcal{L}).
\end{aligned}$$

The last inequality follows from Cauchy–Schwarz inequality. Therefore, we get $I(\mathcal{P},\mathcal{L}) \leq |\mathcal{P}||\mathcal{L}|^{1/2} + |\mathcal{L}|$. By duality of points and lines, namely by the projection that puts points to lines, we also get $I(\mathcal{P},\mathcal{L}) \leq |\mathcal{L}||\mathcal{P}|^{1/2} + |\mathcal{P}|$. These inequalities give us that n points and n lines have $O(n^{3/2})$ incidences. The order $3/2$ can be found in the first chapter, when we examine $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$. The proof we will give is basically the same. Recall that the bound was tight and the construction came from finite fields. On the other hand, in the real plane, $n^{3/2}$ is not tight, as we will see in the next theorem.

Theorem 8.3 (Szemerédi–Trotter). *For any set $\mathcal{P}$ of points and $\mathcal{L}$ of lines in $\mathbb{R}^2$,*

Szemerédi and Trotter (1983)

$$I(\mathcal{P},\mathcal{L}) = O(|\mathcal{P}|^{2/3}|\mathcal{L}|^{3/2} + |\mathcal{P}| + |\mathcal{L}|).$$

Corollary 8.4. *For n points and n lines in $\mathbb{R}^2$, the number of incidences is $O(n^{4/3})$.*

Example 8.5. The bounds in both Theorem 8.3 and Corollary 8.4 are best possible up to a constant factor. Here is an exapmle showing that Corollary 8.4 is tight. Let $\mathcal{P} = [k] \times [2k^2]$ and $\mathcal{L} = \{y = mx + b : m \in [k], b \in [k^2]\}$. Then every line in $\mathcal{L}$ contains k points from $\mathcal{P}$, so $I = k^4 = \Theta(n^{4/3})$.

Proof of Theorem 8.3. we first get rid of all lines in $\mathcal{L}$ which contain at most one point in $\mathcal{P}$. One can see that these lines contribute to at most $|\mathcal{L}|$ incidences.

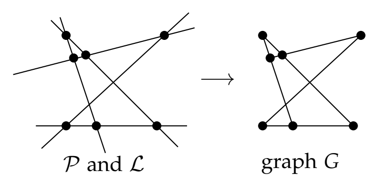

Figure 8.1: Construction of graph G

Now we can assume that every line in $\mathcal{L}$ contains at least two points of $\mathcal{P}$. We construct a graph G as the following: first, we assign vertices to all points in $\mathcal{P}$. For every line in $\mathcal{L}$, we assign an edge between consecutive points of $\mathcal{P}$ lying on the line.

Since a line with k incidences has $k - 1 \geq k/2$ edges, we have the inequality $|E| \geq I(\mathcal{P},\mathcal{L})/2$. If $I(\mathcal{L},\mathcal{P}) \geq 8|\mathcal{P}|$ holds (otherwise, we get $I(\mathcal{P},\mathcal{L}) \lesssim |\mathcal{P}|$), we can apply Theorem 8.2.

$$\mathrm{cr}(G) \gtrsim \frac{|E|^3}{|V|^2} \gtrsim \frac{I(\mathcal{P},\mathcal{L})^3}{|\mathcal{P}|^2}.$$

Moreover $\mathrm{cr}(G) \leq |\mathcal{L}|^2$ since every pair of lines intersect in at most one point. We rearrange and get $I(\mathcal{P},\mathcal{L}) \lesssim |P|^{2/3}|L|^{2/3}$. Therefore we get that $I(\mathcal{P},\mathcal{L}) \lesssim |\mathcal{P}|^{2/3}|\mathcal{L}|^{3/2} + |\mathcal{P}| + |\mathcal{L}|$. The two linear parts are needed for the cases that we excluded in the proof. $\square$

One can notice that we use the topological property of the real plane when we apply Euler's formula in the proof of Theorem 8.2. Now we will present one example of how the sum-product problem is related to incidence geometry.

Theorem 8.6 (Elekes). *If $A \subset \mathbb{R}$, then $|A + A||A \cdot A| \gtrsim |A|^{5/2}$.* Elekes (1997)

Corollary 8.7. *If* $A \subset \mathbb{R}$, *then* $\max\{|A+A|, |A \cdot A|\} \gtrsim |A|^{5/4}$.

Proof of Theorem 8.6. Let $\mathcal{P} = \{(x,y) : x \in A+A, y \in A \cdot A\}$ and $\mathcal{L} = \{y = a(x-a') : a, a' \in A\}$. For a line $y = a(x-a')$ in $\mathcal{L}$, $(a'+b, ab) \in \mathcal{P}$ is on the line for all $b \in A$, so each line in $\mathcal{L}$ contains $|A|$ incidences. By definition of $\mathcal{P}$ and $\mathcal{L}$, we have

$$|\mathcal{P}| = |A+A||A \cdot A| \quad \text{and} \quad |\mathcal{L}| = |A|^2.$$

By Theorem 8.3, we obtain

$$\begin{aligned} |A|^3 \leq I(\mathcal{P}, \mathcal{L}) &\leq |\mathcal{P}|^{3/2}|\mathcal{L}|^{3/2} + |\mathcal{P}| + |\mathcal{L}| \\ &\lesssim |A+A|^{3/2}|A \cdot A|^{3/2}|A|^{4/3}. \end{aligned}$$

Rearranging gives the desired result. □

8.3 *Sum-product via multiplicative energy*

In this chapter, we give a different proof that gives a better lower bound.

Theorem 8.8 (Solymosi)**.** *If* $A \subset \mathbb{R}_{>0}$, *then* Solymosi (2009)

$$|A \cdot A||A+A|^2 \geq \frac{|A|^4}{4\lceil \log_2 |A| \rceil}$$

Corollary 8.9. *If* $A \subset \mathbb{R}$, *then*

$$\max\{|A+A|, |A \cdot A|\} \geq \frac{|A|^{4/3}}{2\lceil \log_2 |A| \rceil^{1/3}}$$

We define ***multiplicative energy*** to be

$$E_\times(A) = |\{(a,b,c,d) \in A^4 : \text{there exists some } \lambda \in \mathbb{R} \text{ such that } (a,b) = \lambda(c,d)\}|$$

Note that the multiplicative energy is a multiplicative version of additive energy. We can see that if A has a small product set, then the multiplicative energy is large.

$$\begin{aligned} E_\times(A) &= \sum_{x \in A \cdot A} |\{(a,b) \in A^2 : ab = x\}|^2 \\ &\geq \frac{|A|^4}{|A \cdot A|} \end{aligned}$$

The inequality follows from Cauchy–Schwarz inequality. Therefore it suffices to show

$$\frac{E_\times(A)}{\lceil \log_2 |A| \rceil} \leq 4|A \cdot A|^2.$$

Proof of Theorem 8.8. We use the dyadic decomposition method in this proof. Let A/A be the set $\{a/b : a, b \in A\}$.

$$
\begin{aligned}
E_\times(A) &= \sum_{s \in A/A} |(s \cdot A) \cap A|^2 \\
&= \sum_{i=0}^{\lceil \log_2 |A| \rceil} \sum_{\substack{s \in A/A \\ 2^i \le ((s \cdot A) \cap A) < 2^{i+1}}} |(s \cdot A) \cap A|^2
\end{aligned}
$$

By pigeonhole principal, there exists some k such that

$$
\frac{E_\times(A)}{\lceil \log_2 |A| \rceil} \le \sum_{\substack{s \in A/A \\ 2^k \le |(s \cdot A) \cap A| < 2^{k+1}}} |(s \cdot A) \cap A|^2.
$$

We denote $D = \{s : 2^k \le |(s \cdot A) \cap A| < 2^{k+1}\}$ and we sort the elements of D as $s_1 < s_2 < \cdots < s_m$. Then one has

$$
\frac{E_\times(A)}{\lceil \log_2 |A| \rceil} \le \sum_{s \in D} |(s \cdot A) \cap A|^2 \le |D| 2^{2k+2}.
$$

For each $i \in [m]$ let ℓ_i be a line $y = s_i x$ and let ℓ_{m+1} be the vertical ray $x = \min(A)$ above ℓ_m.

Let $L_j = (A \times A) \cap \ell_j$, then we have $|L_j + L_{j+1}| = |L_j||L_{j+1}|$. Moreover, the sets $L_j + L_{j+1}$ are disjoint for different j, since they span in disjoint regions.

We can get the lower bound of $|A + A|^2$ by summing up $|L_j + L_{j+1}|$ for all j.

$$
\begin{aligned}
|A + A|^2 &= |A \times A + A \times A| \\
&\ge \sum_{j=1}^{m} |L_j + L_{j+1}| \\
&= \sum_{j=1}^{m} |L_j||L_{j+1}| \\
&\ge m 2^{2k} \ge \frac{E_\times(A)}{4 \lceil \log_2 |A| \rceil}
\end{aligned}
$$

Combining the above inequality with $E_\times(A) \ge |A|^4 / |A \cdot A|$, we reach the conclusion. □

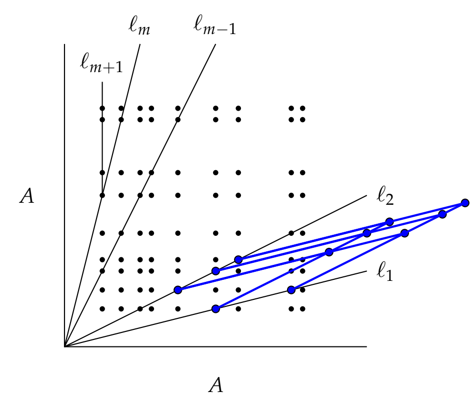

Figure 8.2: Illustration of $L_j + L_{j+1}$