18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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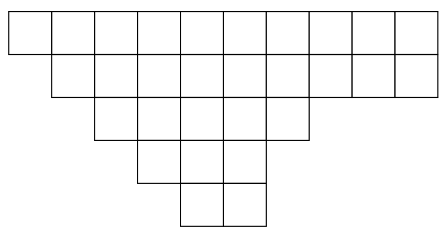
Recall that we discussed Hasse diagrams of posets, and we found that the number of such diagrams (corresponding to linear extensions) for rooted trees is

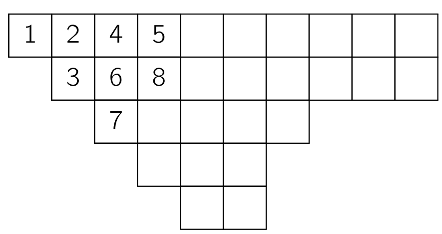
$$\text{ext}(T) = \frac{n!}{\prod_{a \in T} h(a)},$$

where $h(a)$ is the number of nodes in or downstream of a.

Now let's talk about the concept of "shifted Young diagrams." Instead of left-justifying, we can have diagrams like



which corresponds to a partition $\lambda = (10, 9, 5, 3, 2)$. This now gives the number of ways to partition a number n into **distinct** parts! So now we can fill this in the same way: numbers still increase by row and by column:



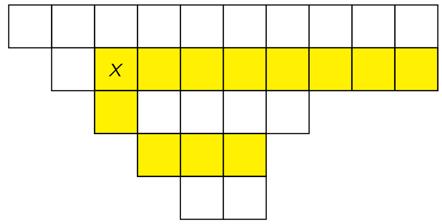
Theorem 1 (Thrall, 1952)

The number of shifted Young tableaux with shifted shape λ with n boxes is similarly

$$\frac{n!}{\prod_{a \in \lambda} h(a)}$$

where $h(a)$, the hook length, now includes a "broken leg."

For example, here's a hook with a broken leg:



Basically, if the hook reaches the left staircase (not the bottom or the right part), it bends over and continues.

The proof of this is similar with the idea of a "hook walk," and it was given by Sagan in 1980.

So the point is that there is a nice number for the number of linear extensions in some posets.

It's time to move on to q-analogs! This stands for both a variable q and for "quantum." The idea is that we can have classical objects and quantum objects, and as we take $q = 1$, we get the classical limit. For example, Planck's constant gives $q = e^{\hbar}$, and as we take $\hbar \to 0$, we basically get $q = 1$.

What are some other examples of this?

Classical	**Quantum**
n	$[n]_q \equiv [n] = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$
$n! = 1 \cdot 2 \cdot \cdots \cdot n$	$[n]_q! \equiv [n]! = [1] \cdot [2] \cdot \cdots \cdot [n]$

So let's look a bit more carefully at what this factorial actually leads to. Binomial coefficients depend on factorials as well!

Classical	**Quantum**
$\binom{n}{k}$	$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]_q!}{[k]_q![n-k]_q!}$

Example 2

In normal numbers, $\binom{4}{2} = 6$. But for the q-factorial,

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4] \cdot [3]}{[1] \cdot [2]} = \frac{(1 + q + q^2 + q^3)(1 + q + q^2)}{(1 + q)} = (1 + q^2)(1 + q + q^2) = 1 + q + 2q^2 + q^3 + q^4.$$

Are there any observations we can make here? If we do some more bashing, we can find the following:

- $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q of the form $a_0 + a_1 q + \cdots + a_m q^m$.
- The coefficients a_i are all positive integers. These are actually called the **Gaussian coefficients.**
- The coefficients are symmetric or **palindromic**: writing them backwards gives back the same thing.
- The coefficients first increase and then decrease: $a_0 \le a_1 \le \cdots \le a_{\lfloor m/2 \rfloor} \ge \cdots \ge a_m$.

Recall that normal coefficients form a Pascal's triangle. Do we have a similar thing here?

$n = 0$: 1

$n = 1$: $1 \quad 1$

$n = 2$: $1 \quad 1 + q \quad 1$

$n = 3$: $1 \quad 1 + q + q^2 \quad 1 + q + q^2 \quad 1$

$n = 4$: $1 \quad 1 + q + q^2 + q^3 \quad 1 + q + 2q^2 + q^3 + q^4 \quad 1 + q + q^2 + q^3 \quad 1$

In normal Pascal's triangle, we have

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}.$$

Turns out something similar occurs here! Notice that each entry of the q-Pascal's triangle is a sum of the two things above it, but one of them is multiplied by a factor of q or something like it.

Proposition 3 (q-Pascal's recurrence relation)

For any n, k,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Proof. We just write out the expressions as factorials:

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \frac{[n-1]!}{[k]![n-k-1]!} + q^{n-k} \frac{[n-1]!}{[k-1]![n-k]!}$$

We can combine common terms:

$$= \frac{[n-1]!}{[k!][n-k]!}([n-k] + q^{n-k}[k])$$

and now note that $[n-k] = 1 + q + q^2 + \cdots + q^{n-k-1}$, and $q^{n-k}[k] = q^{n-k} + \cdots + q^k$. So these work out to just $[n]$, and this gives

$$= \frac{[n-1]!}{[k!][n-k]!}[n] = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

as desired. □

It's time to formulate the combinatorial interpretation for these now! Let λ be a Young diagram, and let $\lambda \subseteq k \times (n-k)$ mean that the Standard (straight) Young diagram fits inside a k by $n-k$ rectangle. So there are at most k nonzero parts, and each one is at most $n-k$. In other words,

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k \geq 0.$$

First of all, the number of standard Young diagrams that fits in here is $\binom{n}{k}$, because we're doing a lattice walk from the bottom left to top right corner!

Theorem 4

For any n, k,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

For example, for $n = 4$, $k = 2$, there are 6 possible Young diagrams. 1 of them has 0 squares, 1 of them has 1 square, 2 of them have 2 squares, 1 of them has 3 squares, and 1 of them has 4 squares!

Note that this immediately implies the first three observations! This shows that the degree of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is $k(n-k)$, and palindromicity comes by taking the complement of the shape for any Young diagram.

There's an interesting way to prove the theorem, but instead, there's always the brute-force method, which is to use induction.

Proof by induction. Base case is not hard to prove. Now, we check the q-Pascal's recurrence for the right hand side.

Look at the first row of the Young diagram. We know $\lambda_1 \leq n-k$.

- If $\lambda_1 < n-k$, which means $\lambda_1 \leq n-k-1$, then λ fits inside a k by $n-k-1$ rectangle instead! This gives the $\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ term.
- Otherwise, $\lambda_1 = n-k$, which means the first row is completely filled. Then if we delete the first row, we get λ inside a $k-1$ by $n-k$ rectangle, and each of these is like an original Young diagram, but we need to add back $n-k$ squares. That's why this contributes a factor of $q^{n-k}\binom{n-1}{k-1}$, and we're done!

$\square$

This doesn't really explain what Young diagrams are coming from, though. Next time, we'll give a more algebraic proof that will explain the source!

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