

## 18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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A few people asked about the problem set. It will be assigned sometime in the near future, and it'll probably be due around the end of February or beginning of March.

Remember that we defined numbers  $f^\lambda$  to be the number of standard Young tableaux of shape  $\lambda$ . These come from the symmetric group, but don't worry about that yet. The magic is the Hook length formula:  $f^\lambda$  has a closed form! Turns out there are some other cool facts along this path.

#### Proposition 1 (Frobenius-Young identity)

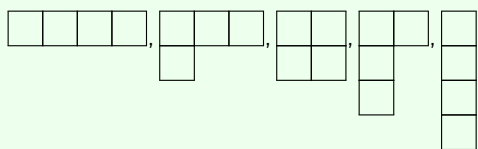
For an integer  $n$ ,

$$\sum_{\lambda: |\lambda|=n} (f^\lambda)^2 = n!.$$

This also comes from representation theory!

#### Example 2

If we take  $n = 4$ , the Young tableaux are



The number of ways to fill in the Young tableaux are 1, 3, 2, 3, 1 respectively, and indeed

$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 24 = 4!.$$

How can we interpret this identity? The right hand side of the equation is the number of permutations in  $S_n$  (on  $n$  elements). Meanwhile, the left hand side counts pairs of Young tableaux  $p, q$ , such that  $p$  and  $q$  are both SYTs of the same shape  $\lambda$ . Is there a nice bijection that we can set up between the two?

Here's the **Schensted correspondence** from 1961, which was generalized later by Knuth to the **Robinson-Schensted-Knuth** correspondence (which is about semi-standard Young tableaux)! See papers on the course webpage. This is a pretty central construction in algebraic combinatorics!

The Schensted's insertion algorithm. Given a standard Young tableaux  $T$  filled with positive integers in  $S$  (for example 1, 3, 7), let's say we want to insert a positive integer  $x$ .

**Definition 3**

Let  $T \leftarrow x$  be the SYT with an extra box added to  $T$ , given by the following procedure:

1. Initialize  $x_1 = x, i = 1$ .
2. If  $x_i$  is greater than all entries in the  $i$ th row of  $T$ , then add a new box at the end of the  $i$ th row filled with  $x_i$  and stop.
3. Otherwise, find the smallest entry  $y$  in the  $i$ th row greater than  $x_i$ . Replace  $y$  by  $x_i$ , increment  $i$  by one, and go back to step 2.

The idea is that we insert  $x$  and bump another entry  $x_2$  down to the next row. Then  $x_2$  will be inserted and bump something else, and so on.

So now given  $w \in S_n$ , let's construct a pair  $(P, Q)$  of Young tableaux.  $P$  is called the **insertion tableau**, and  $Q$  is the **recording tableau**.

To construct  $P$ , start with the empty tableau and add  $w_1, w_2, \dots, w_n$  one at a time, in that order, using the definition above. Meanwhile, to construct  $Q$ ,  $i$  goes to the box that was added at the  $i$ th step of constructing  $P$  (we record which box is added).

**Example 4**

Take the permutation  $w = (3, 5, 2, 4, 7, 1, 6)$ .

After we insert 3 and 5, we get

$$P_0 = \emptyset \implies P_1 = \boxed{3} \implies P_2 = \boxed{3 \mid 5}$$

Trying to insert 2 kicks 3 out into the next row:

$$P_3 = \begin{array}{|c|c|} \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array}$$

Now 4 bumps 5 to get

$$P_4 = \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array}$$

and then adding 7, 1, 6 give

$$P_5 = \begin{array}{|c|c|c|} \hline 2 & 4 & 7 \\ \hline 3 & 5 & \\ \hline \end{array}, P_6 = \begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 5 & \\ \hline 3 & & \\ \hline \end{array}, P_7 = P = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 & & \\ \hline \end{array}$$

Now it is easy to see the  $Q$ -tableau: when were the boxes created?

$$Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline & & 6 \\ \hline \end{array}$$

□

**Theorem 5**

This construction  $w \rightarrow (P, Q)$  is actually a bijection between the symmetric group  $S_n$  and the set of all pairs of Young tableaux with identical shapes and  $n$  boxes!

*Proof.* To show a bijection, we want to say that given a pair  $P$  and  $Q$ , we can reconstruct the permutation. Here's the reverse procedure:

$$P = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 & & \\ \hline \end{array}, Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 & & \\ \hline \end{array}.$$

The position of the 7 in  $Q$  tells us that that box was added last, so the 7 came from the first row. Which entry bumped 7? It must have been 6, the maximal entry of the previous row smaller than 7. So 6 was just added, and that means  $w$  ended with 6. Now we have

$$P' = \begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 5 & \\ \hline 3 & & \\ \hline \end{array}, Q' = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & & \\ \hline \end{array}.$$

Now keep repeating: the bottom left corner has the next maximal entry of  $Q$ , so 3 was bumped last step. It must have been bumped from 2, and that must have been bumped from 1, so  $w$ 's second-to-last entry is 1, and so on. Continuing this procedure, we can recover  $w$ ! So given any pair of tableaux, we can uniquely reconstruct  $w$ , showing the bijection.  $\square$

This correspondence has other special applications as well! The reason was to study increasing and decreasing subsequences of permutations.

### Definition 6

Given a permutation  $w$  that corresponds to a pair  $(P, Q)$  under the Schensted correspondence, the shape  $\lambda$  of  $P$  (and  $Q$ ) is called the **Schensted** shape of  $w$ .

The first entry  $\lambda_1$ , which is the number of boxes in  $\lambda$ , has a special significance:

### Theorem 7 (Schensted)

The length of the first row of  $\lambda$  is the length of the longest increasing subsequence in  $w$ . Meanwhile, as a dual statement, the length of the first column of  $\lambda$  is the length of the longest decreasing subsequence in  $w$ !

For example, the first row has 3 boxes in the example above, so the longest increasing subsequence has length 3. Similarly, the first column has 3 boxes, so the longest decreasing subsequence also has length 3.

We'll wait until next lecture to do this proof.

For now, recall that we proposed previously the following fact:

### Corollary 8

The number of 123-avoiding permutations in  $S_n$  is equal to  $C_n$ .

*Proof.* A permutation  $w$  is 123-avoiding if and only if the length of the longest increasing subsequence is at most 2. Thus, Schensted correspondence sends this to a pair of tableaux with at most 2 columns; both  $P$  and  $Q$  have  $n$  boxes. We want to biject this to a single tableau of shape  $(2, 2, \dots, 2)$ , since we already know that there are  $C_n$  ways to arrange that (bijection to Dyck paths)!

Rotate  $Q$  by 180 degrees and stick it in place with  $P$ , except replace any entry  $k$  with  $(2n + 1) - k$ ! For example,

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 7 \\ \hline 6 & 8 \\ \hline 9 & 10 \\ \hline \end{array}$$

□

Finally, here's a generalization of Schensted. We have an interpretation of  $\lambda_1$ , but we can figure out the whole shape just by looking at the permutation  $w$ !

**Theorem 9 (Green)**

$\lambda_1 + \lambda_2$  is the maximal subsequence that can be covered by two increasing subsequences,  $\lambda_1 + \lambda_2 + \lambda_3$  is the maximal subsequence covered by three subsequences, and so on! The statement also holds for columns and decreasing subsequences.

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