18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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The problem set is posted! It will be due in a week (on May 10). Today, we'll go over some concepts that will be helpful for the problem set. (Again, solving six problems completely is generally good enough.)

Last time, we considered the tree inversion polynomial

$$I_n(x) = \sum_{\substack{T \text{ spanning} \\ \text{trees of } K_{n+1}}} x^{\text{inv}(T)} = \sum_{\substack{f:[n]\to[n] \\ \text{parking function}}} x^{\binom{n+1}{2}-\sum_i f(i)}.$$

Three problems from the problem set are related to alternating permutations $w \in S_n$, which satisfy the condition $w_1 < w_2 > w_3 \cdots$. Letting A_n be the number of alternating permutations of S_n (also known as Euler, Andre, zigzag, tangent and secant numbers, up-down numbers, and so on), these are closely related to the Bernoulli numbers B_n!

> **Fact 1**
>
> Plugging in $x = -1$ yields the Euler numbers A_n.

> **Fact 2**
>
> Taking the **exponential generating function**
> $$\sum_{n \geq 0} A_n \frac{x^n}{n!}$$
> (we do this so that our series is not divergent!), we get $\tan(x) + \sec(x)$. In fact, if we only add odd or even n,
> $$\sum_{n \geq 1 \text{ odd}} A_n \frac{x^n}{n!} = \tan x, \quad \sum_{n \geq 0 \text{ even}} A_n \frac{x^n}{n!} = \sec x.$$

> **Fact 3**
>
> We can draw something similar to the Pascal triangle:

$$
\begin{array}{ccccccccc}
& & & & 1 & & & & \\
& & & 0 & & 1 & & & \\
& & 1 & & 1 & & 0 & & \\
& 0 & & 1 & & 2 & & 2 & \\
5 & & 5 & & 4 & & 2 & & 0
\end{array}
$$

Basically, alternate from left to right, adding up the entries from the row above! Then the nonzero numbers on the sides of the triangle are just A_n.

We can call the right side the "Bernoulli side" and the left side the "Euler side:" the right side is then the numbers that appear in the series expansion of $\tan x$, and the left side is those for $\sec x$.

Definition 4

The **Bernoulli numbers** B_n are given by

$$
\sum_{n \geq 0} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}.
$$

If we expand out, we have

$$
\frac{x}{e^x - 1} = 1 + \left(-\frac{1}{2}\right)x + \frac{1}{6} \cdot \frac{x^2}{2!} + \left(-\frac{1}{30}\right)\frac{x^4}{4!} + \frac{1}{42} \cdot \frac{x^6}{6!} + \cdots
$$

Note that all B_n are only nonzero if n is even or $n = 1$. The exact formula is actually given by (through the $\tan x$ expansion)

$$
B_{2n} = (-1)^{n-1} \frac{2n}{4^{2n} - 2^{2n}} A_{2n-1}.
$$

So really the A_ns are the heart of the combinatorial tricks here!

Fact 5

By the way, these are related to the Riemann zeta function, which can written in integral form as

$$
\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du.
$$

We can find that at integer values, there's a useful relationship:

$$
B_{2n} = \frac{(-1)^{n-1} \cdot 2(2n)!}{(2\pi)^{2n}} \zeta(2n),
$$

which allows us to find $\zeta(2), \zeta(4)$, and so on.

One thing that makes it easier for us to deal with Bernoulli numbers is the useful recurrence relation for A_n:

Proposition 6

We have for all $n \geq 1$ that

$$
2A_{n+1} = \sum_{k=0}^n \binom{n}{k} A_k A_{n-k}.
$$

(Also, we have the initial conditions $A_0 = A_1 = 1$.)

This looks a little like the Catalan number relation (if we remove the 2 and $\binom{n}{k}$). So these are somehow the "labeled" version of the Catalan numbers!

Proof. The left hand side is the number of ways to have alternating permutations on $n + 1$ elements that are either up-down or down-up: $w_1 < w_2 > w_3 \cdots$ or $w_1 > w_2 < w_3 \cdots$.

Take any permutation in this set, and find the position of $(n + 1)$: let's say it's in spot $k + 1$. In front, we have k entries, and after, we have $n - k$ entries: both must be alternating, and the orientation (up-down or down-up) is fixed, because $(n + 1)$ is the largest element. Thus we have $\binom{n}{k}$ ways to pick which k elements are before $n + 1$, $A_k \cdot A_{n-k}$ ways to arrange the remaining elements, and then summing over all k gives the result! □

It turns out that we can turn this recurrence relation to a differential equation for our generating function $A(x)$:

$$2A'(x) + A^2(x) + 1.$$

This is good to look over as an exercise! Note that $A'(x)$ for an exponential generating function shifts the indices by one, and also notice how similar this looks to the Catalan functional equation.

It turns out these ideas are also related to binary trees! There are two versions: a regular **binary tree** is one where every vertex has at most 2 children: a left child and a right child. There's also a notion of a **full binary tree**: this is the subset of normal binary trees where every vertex either has no children (is a leaf) or has both children.

Fact 7

Each full binary tree has an odd number of vertices, because we always add 2 vertices as a time.

It turns out both kinds of binary trees are related to Catalan numbers!

Claim 7.1. The Catalan number C_n is the number of (regular) binary trees on n vertices, and it's also the number of full binary trees on $2n + 1$ vertices.

Proof. It's pretty easy to find a bijection between the sets of trees here: given a binary tree on n vertices, add all missing children (so add both children to each leaf as well). This will create a full binary tree with n non-leafs and $n + 1$ leaves: that means we indeed have a $2n + 1$ vertex full binary tree! To go backwards, just remove all leaves from our tree.

To show that these are equal to C_n, let's find a bijection between Dyck paths of length $2n$ and full binary trees with $2n + 1$ vertices. Given any Dyck path, we can break it up into consecutive up-steps: let $a_1, a_2, \cdots$ be the lengths of those groups. (If we have consecutive down-steps, then we still have $a_i = 0$ in the middle: basically, a_i counts the number of up steps on the ith diagonal.)

Now we can construct our tree: attach a_1 vertices along a left chain (with all children), attach a_2 vertices starting from the bottom-most right child of the a_1-chain, and now repeat. If we have $a_i = 0$, then move one up the diagonal and keep going! This is one way to search through a binary tree. □

Next lecture, we'll discuss what happens when we label the vertices of our binary trees by numbers (with increasing labels downward). Then the number of increasing-labeled binary trees on n nodes is equal to $n!$, and the number of increasing-labeled full binary trees on $2n + 1$ nodes is equal to A_n. So alternating numbers are the "labeled version" of Catalan numbers!

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