18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Today we're going to discuss **Eulerian numbers.**

Definition 1

Let A_{nk} be the number of permutations $w \in S_n$ with exactly k descents: $\text{des}(w) = k$.

Notation is sometimes shifted by one index, but this is not that significant.

Fact 2

It's important not to confuse these with Euler numbers!

There is a nice bijection between permutations in S_n and increasing binary trees on n nodes. Remember that one interpretation of Catalan numbers is the number of binary trees! But this time, we're going to label the vertices by numbers from 1 to n such that they increase as we go away from the root.

Example 3

Start with $w = (4, 2, 8, 5, 1, 3, 9, 10, 6, 7)$.

At the root of this tree, we use 1, and the left branch of the tree contains $(4, 2, 8, 5)$ while the right branch contains everything else. Repeat this process by finding the minimal element in each group, and partition from there!

So each node has at most 1 left child (the smallest element on its left) and at most 1 right child (the smallest element on its right), which means it is indeed a binary tree. The nodes are labeled 1 through n, and they indeed increase downwards.

Fact 4

This is a bijection, since we can just reverse the process by placing 1 in our permutation, putting the left elements to its left and right elements to its right, and so on.

How does this help us say things about Eulerian numbers?

Theorem 5

A_{nk} is the number of increasing binary trees on n nodes with k left edges.

Proof. Think about the bijection. When we go from the permutation to the tree $w \to T_w$, any left edge $i \to j$ means we have

$$w = (\cdots j \cdots) i \cdots,$$

where j is the minimal element in the parentheses. So i will have a descent! This means that the descents correspond to the vertices with a left child, as desired. □

In a sense, Eulerian numbers look a lot like binomial coefficients.

Definition 6

Construct the **Eulerian triangle** so that the bth entry in the ath row is $A_{a,b-1}$. The first few rows look like

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 4 & & 1 & & \\ & 1 & & 11 & & 11 & & 1 & \\ 1 & & 26 & & 66 & & 26 & & 1 \end{array}$$

This is a lot like Pascal's triangle, but the weights are different. If we go along the kth diagonal from either direction, the weight is k! For example, $26 = 4 \cdot 1 + 2 \cdot 11$.

Theorem 7 (Recurrence relation for Eulerian numbers)

For all n, k,

$$A_{n+1,k} = (n-k+1)A_{n,k-1} + (k+1)A_{n,k}.$$

This is not very hard to prove in terms of binary trees or the original descent definition.

Proof. Suppose we have a binary tree with n nodes: we want to add an extra node. We can add an extra leaf with label $n+1$: it can be checked that the number of ways to add a left edge is $n-k+1$, and the number of ways to add a right edge is $k+1$.

Alternatively, we can see this by looking at the permutations: if we have a permutation $w \in S_n$ and we want to add $n+1$. There are $n+1$ ways to add it: adding it in one of the k descent slots or at the beginning does not add a descent, and $k+1$ of these work, while adding it in an ascent slot or at the end adds a descent, which is the other $k-n+1$ ways. □

Remember that last time, we also discussed the two kinds of Stirling numbers. $c(n,k)$, the signless Stirling numbers of the first kind, is the number of permutations $w \in S_n$ with k cycles (including fixed points). Make a Pascal's triangle such that the bth entry in the ath row is $c(a,b)$. Then this triangle looks like

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 2 & & 3 & & 1 & \\
6 & & 11 & & 6 & & 1
\end{array}
$$

Notice that we no longer have the symmetry between k cycles and $n-k$ cycles. But we still have a recurrence relation with weights!

Proposition 8 (Recurrence relation for $c(n,k)$)
For all n, k,

$$c(n+1,k) = c(n,k-1) + nc(n,k).$$

Meanwhile, $S(n,k)$, the Stirling numbers of the second kind, are the number of set-partitions of $[n]$ into k groups. If we construct a similar triangle, it looks like

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 3 & & 1 & & \\
 & 1 & & 7 & & 6 & & 1 & \\
1 & & 15 & & 25 & & 10 & & 1
\end{array}
$$

It is a bit harder to describe the weights here: they are 1 for all edges sloping down and to the right, and they increase starting from 1 in each row for edges sloping down and to the left.

Proposition 9 (Recurrence relation for $S(n,k)$)
For all n, k,

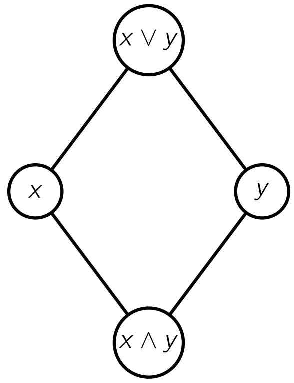
$$S(n+1,k) = S(n,k-1) + kS(n,k).$$

As an exercise, we should prove these two relations!

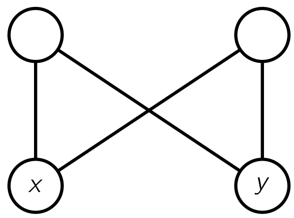
There are many more permutation statistics, but we'll move on to the next topic for now: posets and lattices!

Definition 10
A **lattice** L is a special kind of poset: it has two binary operations **meet** $\wedge$ and **join** $\vee$. $x \wedge y$ is the unique maximal element of L which is less than or equal to both x and y, and $x \vee y$ is the unique minimal element of L such that it is greater than or equal to both x and y.

Here's an example of a lattice:



However, here's an example of something that is not a lattice, since x and y have no $x \wedge y$ and two different potential $x \vee y$s:



The usual definition doesn't actually use posets though: it's more abstract.

Definition 11 (Axiomatic definition of a lattice)

A **lattice** is a set L with binary operations $\wedge$ and $\vee$ such that

- $\wedge$ and $\vee$ are commutative and associative, so $x \wedge y = y \wedge x$, and $x \vee y = y \vee x$. Also, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, and similar for $\vee$.
- $x \wedge x = x \vee x = x$.
- (Absorption law) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.
- $x \wedge y = x$ if and only if $x \vee y = y$.

  Note that addition and multiplication do not follow these laws.

Fact 12

Given the axiomatic operations and a set L, we can reconstruct a poset, and given any lattice, it and its operations $\wedge$ and $\vee$ satisfy the axioms! In particular, $x \leq y$ happens if and only if $x \wedge y = x$, which is the same as $x \vee y = y$.

Some time in the future, we will discuss boolean lattices, partition lattices, and Young lattices!