Lecture 4

Removable Singularity Theorem

Theorem 1 *Let u be harmonic in $\Omega \setminus \{x_0\}$, if*

$$u(x) = \begin{cases} o(|x-x_0|^{2-n}) & , \quad n > 2, \\ o(\ln|x-x_0|) & , \quad n = 2 \end{cases}$$

as $x \to x_0$, then u extends to a harmonic function in Ω.

Proof: Without loss of generality, we can assume $\Omega = B(0,2)$, then $u|_{\partial B(0,1)}$ is continuous. Thus by Poisson Integral formula, $\exists v \in C(\overline{B(0,1)}) \cap C^\infty(B(0,1))$ to be harmonic function with boundary condition $v = u$ on $\partial B(0,1)$.

Choose $\epsilon > 0$ and $\delta > 0$ small, consider

$$\omega(x) = \begin{cases} u(x) - v(x) - \epsilon(|x|^{2-n} - 1) & , \quad n > 2, \\ u(x) - v(x) + \epsilon \log|x|) & , \quad n = 2, \end{cases}$$

then $\omega(x)$ is harmonic on $B_1(0) \setminus BB_\delta(0)$, and $\omega(x) = 0$ on $\partial B_1(0)$.

On $\partial B_\delta(0)$, $-\epsilon|x|^{2-n}$ is the dominate term, thus $\omega \le 0$ on $\partial B_\delta(0)$ for δ small enough.

Now by maximum principle, $\omega \le 0$ on $B_1(0) \setminus B_\delta(0)$, i.e.

$$u(x) \le v(x) + \epsilon(|x|^{2-n} - 1),$$

Thus by letting $\epsilon \to 0$, we get

$$u(x) \le v(x), \forall x \in B_1(0) \setminus B_\delta(0).$$

This is true for any δ small, so it is true for $\forall x \in B_1(0) \setminus \{0\}$.

By reverting u and v, we can get

$$v(x) \le u(x), \forall x \in B_1(0) \setminus \{0\},$$

thus $v(x) = u(x), \forall x \in B_1(0) \setminus \{0\}$.

Now we can define $u(0) = v(0)$, and extend u to be a harmonic function on $B(0,1)$, thus a harmonic function on $\Omega = B(0,2)$. ■

Example This gives an example of Dirichlet problem that is **NOT** solvable:

Take $\Omega = B(0,1) \setminus \{0\}$, then $\partial\Omega = \partial B(0,1) \cup \{0\}$. Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 & , \quad in \ \ \Omega, \\ u = 0 & , \quad on \ \ \partial B(0,1), \\ u = 1 & , \quad at \ \ 0 \end{cases}$$

If this is solvable, then the solution u can be extend to a bounded harmonic function on $B(0,1)$. Now by MVP, $u(0)=0$, which is a contradiction.

Laplacian in general coordinate systems

Theorem 2 *Let g_{ij} be the metric component of a coordinate system, then*

$$\Delta u = \frac{1}{\sqrt{det(g_{rs})}}\partial_k(g^{kj}\partial_j u\sqrt{det(g_{rs})}).$$

Proof: Take any $\varphi \in C_0^\infty$, we have

$$\begin{aligned}
\int \varphi\Delta u\sqrt{det(g_{ij})}dy &= \int \varphi\Delta u dx \\
&= \int < \nabla\varphi, \nabla u > dx \\
&= \int g^{ij}\partial_i\varphi\partial_j u\sqrt{det(g_{ij})}dy \\
&= \int \varphi\partial_i(g^{ij}\partial_j u\sqrt{det(g_{ij})})dy \\
&= \int \varphi\sqrt{det(g_{ij})}\frac{\partial_i(g^{ij}\partial_j u\sqrt{det(g_{ij})})}{\sqrt{det(g_{ij})}}dy.
\end{aligned}$$

Thus the formula follows. ■

Laplacian in spherical coordinates (r,ω)

Now $g = dr^2 + r^2 g_{S^{n-1}}$, so

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 g_{S^{n-1}} \end{pmatrix} \implies (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} g^{ij}_{S^{n-1}} \end{pmatrix}.$$

so

$$\sqrt{det(g_{ij})} = \sqrt{r^{2(n-1)}det(g_{S^{n-1}})} = r^{n-1}\sqrt{det(g_{S^{n-1}})},$$

thus

$$\begin{aligned}
\Delta u &= \frac{1}{r^{n-1}\sqrt{det(g_{S^{n-1}})}}\partial_1(g^{1j}\partial_j u\ r^{n-1}\sqrt{det(g_{S^{n-1}})}) \\
&\quad + \frac{1}{r^{n-1}\sqrt{det(g_{S^{n-1}})}}\sum_{k>1}\partial_k(g^{kj}\partial_j u\ r^{n-1}\sqrt{det(g_{S^{n-1}})}) \\
&= \frac{1}{r^{n-1}}\partial_r(\partial_r u\ r^{n-1}) + \frac{1}{\sqrt{det(g_{S^{n-1}})}}\sum_{k>1}\partial_k(r^{-2}g^{ij}_{S^{n-1}}\partial_j u\sqrt{det(g_{S^{n-1}})}) \\
&= \frac{1}{r^{n-1}}\partial_r(\partial_r u\ r^{n-1}) + r^{-2}\Delta_{S^{n-1}}u \\
&= \frac{1}{r^{n-1}}(u_{rr}\ r^{n-1} + u_r(n-1)r^{n-2}) + r^{-2}\Delta_{S^{n-1}}u \\
&= u_{rr} + (n-1)\frac{u_r}{r} + \frac{1}{r^2}\Delta_{S^{n-1}}u.
\end{aligned}$$

If $u(r,\theta) = f(r)B(\theta)$ is variables separated, then

$$\Delta u(r,\theta) = (f_{rr} + (n-1)\frac{f_r}{r})B_\theta + \frac{f(r)}{r^2}\Delta_{S^{n-1}}B(\theta).$$

Proposition 1 *Let $B(\theta)$ be a homogeneous harmonic polynomial of degree k restricted to S^{n-1}, then $\Delta_{S^{n-1}}B(\theta) = -k(k+n-2)B(\theta)$.*

Remark 1 *Let $\mathcal{P}_k$ be the set of homogeneous polynomials of degree k on $\mathbb{R}^n$, $\mathcal{H}_k$ be the set of harmonic homogeneous polynomials of degree k on $\mathbb{R}^n$, then*

$$\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{P}_{k-2}.$$

It's not hard to prove

$$dim\mathcal{P}_k = \frac{(k+n-1)!}{k!(n-1)!},$$

so

$$dim\mathcal{H}_k = \frac{(k+n-1)!}{k!(n-1)!} - \frac{(k+n-3)!}{(k-2)!(n-1)!} = (2k+n-2)\frac{(k+n-3)!}{k!(n-2)!}.$$

For such a $B(\theta) \in \mathcal{H}_k$, we have

$$\Delta(f(r)B(\theta)) = (f_{rr} + \frac{n-1}{r}f_r - k(k+n-2)\frac{f_r}{r^2})B(\theta).$$

For the solution of the equation

$$f_{rr} + \frac{n-1}{r}f_r - k(k+n-2)\frac{f_r}{r^2} = 0,$$

let $f = r^p$, then $f_r = p\ r^{p-1}$, $f_{rr} = p(p-1)r^{p-2}$, we get

$$0 = p(p-1)r^{p-2} + p(n-2)r^{p-2} - k(k+n-2)r^{p-2} = (p-k)(p+k+n-2)r^{p-2}.$$

Thus $p = k$ or $p = -k - n + 2$.

For $p = k$, we get $u(r, \theta) = r^k B(\theta)$, where $B(\theta) \in \mathcal{H}_k$, thus u is just the homogeneous k harmonic polynomial on $\mathbb{R}^n$.

For those $p = -k - n + 2$, if $k = 0$, then $p = 2 - n$ and $B(\theta) = constant$, thus $u = c \cdot r^{2-n}$, which is the fundamental solution. if $k > 0$, then $p < 2 - n$, note that $B(\theta)$ is defined on the compact set S^{n-1}, thus B is bounded, so u grows faster than the fundamental solution near the origin.

From above we get a degree gap of harmonic function:

$$\cdots\cdots, -n, -(n-1), -(n-2), \blacksquare, 0, 1, 2, \cdots\cdots$$

Notice that we have to have the gap in view of our removable singularity theorem.

Homogeneous expansions

Theorem 3 *Any harmonic function in $B(0,1)$ can be expressed as an infinite sum*

$$u(x) = \sum_{k=0}^{\infty} p_k(x), \qquad p_k \in \mathcal{H}_k.$$

Proof: Take the Taylor expansion of u, $u = \sum p_k$, where $p_k \in \mathcal{P}_k$, we have

$$0 = \Delta u = \sum \Delta p_k,$$

but $\Delta p_k \in \mathcal{P}_{k-2}$, thus $\Delta p_k = 0$ for all k, i.e. $p_k \in \mathcal{H}_k$. $\blacksquare$