Lecture 20

April 29th, 2004

Difference Quotients and Sobolev spaces

$\mathcal{D}$efine

$$\Delta_i^h u := \frac{u(x + h \cdot \mathbf{e_l}) - u(x)}{h}, \quad h \neq 0.$$

$\mathcal{L}$emma. *Let Ω be a bounded domain in $\mathbb{R}^n$, and $u \in W^{1,p}(\Omega)$, for some $1 \leq p < \infty$. Then for any $\Omega' \Subset \Omega$ such that $dist(\Omega', \partial\Omega) > h$ holds*

$$||\Delta_i^h u||_{L^p(\Omega')} \leq ||D_i u||_{L^p(\Omega)}.$$

Proof.

$$|\Delta_i^h u| = \Big|\frac{u(x + h \cdot \mathbf{e_l}) - u(x)}{h}\Big| \leq \frac{1}{h} \int_0^h \big|\mathrm{D}_i u(x_1, \ldots, x_i + \zeta, \ldots, x_n)\big| d\zeta$$

$$\leq \frac{1}{h} \Big\{\int_0^h 1^q\Big\}^{\frac{1}{q}} \Big\{\int_0^h \big|\mathrm{D}_i u(x_1, \ldots, x_i + \zeta, \ldots, x_n)\big|^p d\zeta\Big\}^{\frac{1}{p}} \quad \Rightarrow$$

$$|\Delta_i^h u|^p \leq h^{\frac{p}{q} - p} \cdot \int_0^h \big|\mathrm{D}_i u(x_1, \ldots, x_i + \zeta, \ldots, x_n)\big|^p d\zeta$$

$$= \frac{1}{h} \cdot \int_0^h \big|\mathrm{D}_i u(x_1, \ldots, x_i + \zeta, \ldots, x_n)\big|^p d\zeta \quad \Rightarrow$$

$$\int_{\Omega'} |\Delta_i^h u|^p \leq \frac{1}{h} \cdot \int_{\Omega'} \int_0^h \big|\mathrm{D}_i\big|^p d\zeta d\mathbf{x} = \frac{1}{h} \cdot \int_0^h \int_{\Omega}^{\prime} \big|\mathrm{D}_i\big|^p d\mathbf{x} d\zeta$$

$$= \frac{1}{h} \int_0^h ||\mathrm{D}_i u||_{L^p(\Omega')} = ||\mathrm{D}_i u||_{L^p(\Omega')} \leq ||\mathrm{D}_i u||_{L^p(\Omega)},$$

where we applied Fubini's Theorem in order to switch order of integration. ■

Conversely we have

Lemma. *Let $u \in L^p(\Omega)$ for some $1 \leq p < \infty$ and suppose $\Delta_i^h u \in L^p(\Omega')$ with $||\Delta_i^h u||_{L^p(\Omega')} \leq K$ for all $\Omega' \Subset \Omega$ and $0 < |h| < dist(\Omega', \Omega)$. Then the weak derivative satisfies $||D_i u||_{L^p(\Omega)} \leq K$. Consequently if this holds for all $i = 1, \ldots, n$ then $u \in W^{1,p}(\Omega)$.*

Proof. We will make use of

Alouglou's Theorem. *A bounded sequence in a separable, reflexive Banach space has a weakly convergent subsequence.*

A topological space is called *separable* if it contains a countable dense set.

A Banach space is called *reflexive* if $(B^\star)^\star = B$.

A sequence $\{x_n\}$ in a Banach space is said to *converge weakly* to x when $\lim_{n\to\infty} F(x_n) \to F(x)$ for all linear functionals $F \in B^\star$. This is sometimes denoted $\lim_{n\to\infty} x_n \rightharpoonup x$.

Example: Let $\ell^2 := \left\{(a_1, a_2, \ldots) : \sum_{i=1}^{\infty} a_i^2 < \infty\right\}$. Consider the sequence $\{x_i := (0, \ldots, 0, 1, 0, \ldots)\} \subseteq \ell^2$. Any bounded linear functional on ℓ^2 will be some linear combination of the linear functionals F_j, defined by $F_j(a_1, \ldots) = a_j$ (each such linear combination corresponds exactly to a point in ℓ^2. That makes sense, indeed by the Riesz Representation Theorem $(\ell^2)^\star = \ell^2$ (note ℓ^2 is a Hilbert space not just a Banach space as it has an inner product structure).). For any such $F = (a_1, \ldots)$, $\lim_{i\to\infty} F(x_i) = \lim_{i\to\infty} a_i = 0$. So x_i converges to the 0 vector weakly, though certainly not strongly: by Fourier Theory each point in ℓ^2 corresponds to a periodic function on $[0, 1]$, i.e an element of $L^2(S^1)$, and of course $\lim_{n\to\infty} \exp(n2\pi\sqrt{-1}z) \not\to 0(z)$.

We come back to the proof. For the Banach space $B = L^p(\Omega)$, $B^\star = L^q(\Omega)$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. This can be seen directly: If $F \in (L^p(\Omega))^\star$, then exists f such that $F(g) = \int_\Omega g \cdot f, \quad \forall g \in L^p(\Omega)$, and this will be bounded iff $f \in L^q(\Omega)$. So we get an identification $F \in (L^p(\Omega))^\star \cong L^q(\Omega)$.

By Alouglou's Theorem there exists a sequence $\{h_m\} \to 0$ with $\Delta_i^{h_m} u \rightharpoonup v \in L^p(\Omega)$. In other words

$$\int_\Omega \psi \cdot \Delta_i^{h_m} u \to \int_\Omega \psi \cdot v \in L^p(\Omega), \quad \forall\, \psi \in L^q(\Omega).$$

And in particular for any $\psi \in \mathcal{C}_0^1(\Omega)$ (which is dense in $L^q(\Omega)$ so will suffice to look at such ψ as will become clear ahead)

$$\begin{aligned}
\int_\Omega \psi \Delta_i^{h_m} u &= \int_\Omega \psi \frac{1}{h}(u(x + h \cdot \mathbf{e_l}) - u(x))d\mathbf{x} \\
&= \frac{1}{h}\int_\Omega \psi(x - h\mathbf{e_i})u(x)d\mathbf{x} - \frac{1}{h}\int_\Omega \psi(x)u(x)d\mathbf{x} \\
&= \int_\Omega \frac{1}{h}(\psi(x - h\mathbf{e_i}) - \psi(x))u(x)d\mathbf{x} \\
&= \int_\Omega -\Delta_i^h \psi(x)u(x)d\mathbf{x} \quad \overset{h\to 0}{\longrightarrow} \quad \int_\Omega -\mathrm{D}_i\psi(x)u(x)d\mathbf{x}
\end{aligned}$$

since ψ is continuously differetiable. Altogether

$$\int_\Omega \psi \cdot v \in L^p(\Omega) = \int_\Omega -\mathrm{D}_i\psi(x)u(x)d\mathbf{x},$$

which by definition means v is the weak derivative of u in the direction of the x_i axis, or simply the undistinctive notation $v = \mathrm{D}_i u$.

We also get the desired estimate, using the Fatou Lemma $\int \liminf \leq \liminf \int$

$$\int_\Omega |\mathrm{D}_i u|^p d\mathbf{x} = \int_\Omega \liminf |\Delta_i^h u|^p d\mathbf{x} \leq \liminf \int_\Omega |\Delta_i^h u|^p d\mathbf{x} \leq K^p,$$

i.e $||\mathrm{D}_i u||_{L^p(\Omega)} \leq K$. ∎

$\mathbf{L^2}$ Theory

$\mathcal{C}$onsider the second order equation in divergence form

$$Lu \equiv L(u) := \mathrm{D}_i(a^{ij}\mathrm{D}_j u) + b^i\mathrm{D}_i u + c \cdot u = f,$$

with $a^{ij}, b^i, c \in L^1(\Omega)$ (integrable coefficients).

We call $u \in W^{1,2}(\Omega)$ a *weak solution* of the equation if

$$\forall \ \ v \in \mathcal{C}_0^1(\Omega) \qquad -\int_\Omega a^{ij}\mathrm{D}_j u \mathrm{D}_i v + \int_\Omega (b^i \mathrm{D}_i u + cu)v = \int_\Omega fv.$$

Elliptic Regularity

Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, and assume

- *L strictly elliptic with $(a^{ij}) > \gamma \cdot I, \ \ \gamma > 0$*
- $a^{ij} \in \mathcal{C}^{0,1}(\Omega)$
- $b^i, c \in L^\infty(\Omega)$
- $f \in L^2(\Omega)$

Then for any $\Omega' \Subset \Omega, \ \ u \in W^{2,2}(\Omega')$ and

$$||u||_{W^{2,2}(\Omega')} \leq C(||a^{ij}||_{C^{0,1}(\Omega)}, ||b||_{C^0(\Omega)}, ||c||_{C^0(\Omega)}, \lambda, \Omega', \Omega, n) \cdot \big(||u||_{W^{1,2}(\Omega)} + ||f||_{L^2(\Omega)}\big).$$

Note $L^\infty(\Omega)$ stands for bounded functions on Ω while $\mathcal{C}^0(\Omega)$ are functions that are also continuous (Ω being bounded).

Proof. Start with the definition of u being a solution in the weak sense, $\forall \ \ v \in \mathcal{C}_0^1(\Omega)$:

$$\int_\Omega a^{ij} \mathrm{D}_j u \mathrm{D}_i v = \int_\Omega (b^i \mathrm{D}_i u + c - f)v.$$

and take difference quotients, that is replace v with $\Delta^{-h} v$.

$$\int_\Omega a^{ij} \mathrm{D}_j u \mathrm{D}_i (\Delta^{-h} v) = \int_\Omega (b^i \mathrm{D}_i u + c - f)(\Delta^{-h} v).$$

Taking $-h$ is a technicality that will unravel its reason later on, and we really mean $\Delta_k^{-h} v$ for some $k \in \{1, \ldots, n\}$ and then eventually repeat the computation for all k in that range. This will become clear as well. Finally our goal will be to use the Chain Rule and move the difference quotient operator onto u under the integral sign and get uniform bounds on $\Delta^h \mathrm{D} u$ and in this way get a priori $W^{2,2}(\Omega)$ estimates.

The Chain Rule gives

$$\Delta^h (a^{ij} \mathrm{D}_j u) =$$

$$\frac{1}{h}\big(a^{ij} u(x + h \cdot \mathbf{e_k}) \mathrm{D}_j u(x + h \cdot \mathbf{e_k}) - \{a^{ij}(x) - a^{ij}(x + h \cdot \mathbf{e_k}) + a^{ij}(x + h \cdot \mathbf{e_k})\} \mathrm{D}_j u(x)\big)$$

$$= a^{ij} u(x + h \cdot \mathbf{e_k}) \Delta^h \mathrm{D}_j u - \Delta^h a^{ij} \mathrm{D}_j u.$$

And applied to our previous equation, a short calculation verifies that we can 'integrate by part' WRT Δ^h–

$$\int_\Omega a^{ij} \mathrm{D}_j u \mathrm{D}_i (\Delta^{-h} v) = \int_\Omega \Delta^h (a^{ij} \mathrm{D}_j u) \mathrm{D}_i v \quad \Rightarrow$$

$$\int_\Omega a^{ij} u(x + h \cdot \mathbf{e_k}) \Delta^h \mathrm{D}_j u \mathrm{D}_i v = \int_\Omega -\Delta^h a^{ij} \mathrm{D}_j u \mathrm{D}_i v + \int_\Omega (b^i \mathrm{D}_i u + c - f)(\Delta^{-h} v) \quad \Rightarrow$$

$$\Big| \int_\Omega a^{ij} u(x + h \cdot \mathbf{e_k}) \Delta^h \mathrm{D}_j u \mathrm{D}_i v \Big| \leq ||\Delta^h a^{ij} \mathrm{D}_j u||_{L^2(\Omega)} ||\mathrm{D}_i v||_{L^2(\Omega)} +$$

$$+ ||b^i \mathrm{D}_i u + cu - f||_{L^2(\Omega)} ||\Delta^{-h} v||_{L^2(\Omega)},$$

where we have used the Hölder Inequality for $p = q = 2$. This in turn can be bounded by

$$\leq ||a^{ij}||_{C^{0,1}(\Omega)}||\mathrm{D}u||_{L^2(\Omega)}||\mathrm{D}v||_{L^2(\Omega)}+$$

$$+\big(||b^i||_{L^\infty(\Omega)}||\mathrm{D}u||_{L^2(\Omega)}+||c||_{L^\infty(\Omega)}||u||_{L^2(\Omega)}+||f||_{L^2(\Omega)}\big)||\mathrm{D}v||_{L^2(\Omega)}$$

$$\leq C(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)})\cdot||\mathrm{D}v||_{L^2(\Omega)}.$$

where we have used the Hölder Inequality for $p = 1, q = \infty$, i.e a simple bounded integration argument (e.g $||cu||_{L^2(\Omega)} = \big(\int c^2 \cdot |u|^2\big)^{\frac{1}{2}} \leq \big(\sup |c|^2 \int_O |u|^2\big)^{\frac{1}{2}}$), and $\Delta^h a^{ij} \to \mathrm{D}_k a^{ij}$ as $a^{ij}\mathcal{C}^{0,1}(\Omega)$.

Take a cut-off function $\eta \in \mathcal{C}_0^1(\Omega), 0 \leq |\eta| \leq 1, \eta\big|_{\Omega'} = 1$. We now choose a special v: $v := \eta^2\Delta^h u$. From uniform ellipticity ($a^{ij}\zeta_i\zeta_j \geq \lambda|\zeta|^2$)

$$\lambda\int_\Omega |\eta\mathrm{D}\Delta^h u|^2 \leq \int_\Omega \eta^2 a^{ij}(x+h\cdot\mathbf{e_k})\mathrm{D}_i\Delta^h u\mathrm{D}_j\Delta^h u.$$

Now

$$\mathrm{D}_i v = 2\eta\mathrm{D}_i\eta\Delta^h u + \eta^2\mathrm{D}_i\Delta^h u$$

which we substitute into our previous inequality,

$$\int_\Omega \eta^2 a^{ij}(x+h\cdot\mathbf{e_k})\mathrm{D}_j\Delta^h u\mathrm{D}_j\Delta^h u \leq \int_\Omega a^{ij}(x+h\cdot\mathbf{e_k})\mathrm{D}_j\Delta^h u\cdot(\mathrm{D}_i v - 2\eta\mathrm{D}_i\eta\Delta^h u)$$

$$\leq C(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)})||\mathrm{D}v||_{L^2(\Omega)}+$$

$$+\,C'||(\mathrm{D}\Delta^h u)\eta||_{L^2(\Omega)}||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega)}$$

again by the Hölder Inequality. Now since $\eta \leq 1$

$$||\mathrm{D}_i v||_{L^2(\Omega)} \leq C''\big(||\mathrm{D}_i\eta\Delta^h u||_{L^2(\Omega)}+||\mathrm{D}\Delta^h u||_{L^2(\Omega)}\big).$$

Combining all the above and again using $\eta \leq 1$,

$$\lambda\int_{\Omega}^{\prime}|\eta \mathrm{D}\Delta^h u|^2 \leq C\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}\big)\cdot C''\big(||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega')}+||\mathrm{D}\Delta^h u||_{L^2(\Omega')}\big)$$

$$+C'||(\mathrm{D}\Delta^h u)||_{L^2(\Omega')}||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega')}$$

$$\leq c\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}+||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega')}\big)\cdot||(\mathrm{D}\Delta^h u)||_{L^2(\Omega')}$$

$$+c\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}\big)\cdot||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega')}.$$

Using the AM-GM Inequality $ab=\sqrt{\frac{1}{\epsilon}a^2\cdot\epsilon b^2}\leq\frac{1}{2}\big(\frac{1}{\epsilon}a^2+\epsilon b^2\big)$ for the first term and the inequality $(a+b)c\leq\frac{1}{2}(a+b+c)^2$ for the second

$$\lambda\int_{\Omega'}|\eta\mathrm{D}\Delta^h u|^2 \;\leq\; \frac{1}{\epsilon}c^2\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}+||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega')}\big)^2+\epsilon||(\mathrm{D}\Delta^h u)||^2_{L^2(\Omega')}$$

$$+c\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}+||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega')}\big)^2.$$

Choose any $0<\epsilon<\lambda/2$. Then subtract the second term on the first line of the RHS from the LHS to get

$$||\eta\mathrm{D}\Delta^h u||^2_{L^2(\Omega')} \;\leq\; c\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}+||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega')}\big)^2 \quad\Rightarrow$$

$$||\eta\mathrm{D}\Delta^h u||_{L^2(\Omega')} \;\leq\; c\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}+||\mathrm{D}\eta\Delta^h u||_{L^2(\Omega')}\big)$$

$$\leq c\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}+\sup_{\Omega}|\mathrm{D}\eta|\cdot||\Delta^h u||_{L^2(\Omega')}\big)$$

$$\leq c\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}\big)\cdot\big(1+\sup_{\Omega}|\mathrm{D}\eta|\big),$$

since $||\Delta^h u||_{L^2(\Omega)}\leq||\mathrm{D}u||_{L^2(\Omega)}\leq||u||_{W^{1,2}(\Omega)}\leq||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}$ where we have applied the first Lemma to $u\in W^{1,2}(\Omega)$. Now we are done as we can choose η such that first $\eta\big|_{\Omega'}=1$ (for the LHS !) and second $|\mathrm{D}\eta|\leq\mathrm{dist}(\Omega',\partial\Omega)$ (for the RHS) and so

$$||1\cdot\mathrm{D}\Delta^h u||^2_{L^2(\Omega')} \;\leq\; c\big(||u||_{W^{1,2}(\Omega)}+||f||_{L^2(\Omega)}\big),$$

independently of h. So by our second Lemma the uniform boundedness of the difference quotients of $\mathrm{D}u$ in $L^2(\Omega')$ implies $\mathrm{D}u \in W^{1,2}(\Omega') \quad \Rightarrow \quad u \in W^{2,2}(\Omega')$ and we have the desired estimate for its $W^{2,2}(\Omega')$ norm by the above inequality combined with the Lemma. ∎

Now that $u \in W^{2,2}(\Omega')$ then the our original equation holds in the *usual* sense

$$Lu = a^{ij}\mathrm{D}_i j u + \mathrm{D}_i a^{ij}\mathrm{D}_j u + b^i \mathrm{D}_i u + c \cdot u = f,$$

a.e !