Lecture 19

April 27th, 2004

We give a slightly different proof of

Theorem. *Let Ω a bounded domain in $\mathbb{R}^n$, and $1 \leq p < \infty$.*

$$W_0^{1,p}(\Omega) \subseteq \mathcal{C}^{0,\alpha}(\Omega), \quad \alpha = 1 - \frac{n}{p}, \quad p > n,$$

and $\exists\, C(n, p, \Omega)$ such that for $u \in W_0^{1,p}(\Omega)$

$$||u||_{C^{0,\alpha}(\Omega)} \leq C \cdot ||u||_{W^{1,p}(\Omega)}, \quad \forall p > n,$$

in other words

$$\sup_{\Omega} |u| + |u|_{C^{0,\alpha}(\Omega)} \leq C \cdot \left\{ ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)} \right\}, \quad \forall p > n.$$

Note the inequality is stronger than the one we stated in the previous lecture.

Proof. We take $u \in \mathcal{C}_0^1(\Omega)$ as before, WLOG (density argument). Extend u to $\mathbb{R}^n$ trivially, i.e set $u = 0$ on $\mathbb{R}^n \setminus \Omega$. Let $x, y \in \Omega$ and $\sigma = |x - y|$ and let p be the point $\dfrac{x+y}{2}$. Put $B = B(p, \sigma)$ and take $z \in B$. By the Fundamental Theorem of Calculus

$$\begin{aligned} u(x) - u(z) &= \int_0^1 \frac{d}{dt} u(x + (1-t)z) dt \\ &= \int_0^1 \nabla u(x + t(z - x)) \cdot (z - x)\, dt. \end{aligned}$$

Integrating over $z \in B$

$$\Big|\int_B u(z)d\mathbf{z} - \mathrm{Vol}(B)u(x)\Big| \le \int_B \int_0^1 |\nabla u(x+t(z-x))|\cdot|z-x|dtd\mathbf{z}$$

$$\le 2\sigma \int_B \int_0^1 |\nabla u(x+t(z-x))|dtd\mathbf{z}$$

$$= 2\sigma \int_0^1 \Big(\int_B |\nabla u(x+t(z-x))|d\mathbf{z}\Big)dt.$$

Change variables to

$$\bar{z} := x + t(z-x), \quad \to \quad d\bar{\mathbf{z}} = t^n d\mathbf{z}.$$

For $z \in B(x,\sigma) \;\Rightarrow\; \bar{z} \in B(x,t\sigma) =: \bar{B}$. In the new variable we have now

$$\Big|\int_B ud\mathbf{z} - \mathrm{Vol}(B)u(x)\Big| \le 2\sigma \int_0^1 t^{-n}\Big(\int_{\bar{B}} |\nabla u(\bar{z})|d\bar{\mathbf{z}}\Big)dt.$$

By the Hölder Inequality for q such that $\dfrac{1}{p}+\dfrac{1}{q}=1$

$$\int_{\bar{B}} |\nabla u(\bar{z})|d\bar{\mathbf{z}}\Big)dt \le \Big\{\int_{\bar{B}} 1^q\Big\}^{\frac{1}{q}} \cdot \Big\{\int_{\bar{B}} |\nabla u(w)|^p d\mathbf{w}\Big\}^{\frac{1}{p}}$$

$$= \mathrm{Vol}(B(t\sigma))^{\frac{1}{q}}||\nabla u||_{L^p(\bar{B})}$$

$$\le \mathrm{Vol}(B(t\sigma))^{\frac{1}{q}}||\nabla u||_{L^p(\Omega)}$$

$$= \omega_n^{\frac{1}{q}} t^{\frac{n}{q}} \sigma^{\frac{n}{q}} ||\nabla u||_{L^p(\Omega)} \qquad \Rightarrow$$

$$\Big|\int_B ud\mathbf{z} - \mathrm{Vol}(B)u(x)\Big| \le 2\sigma^{1+\frac{n}{q}}\omega_n^{\frac{1}{q}}\Big(\int_0^1 t^{-n}\cdot t^{\frac{n}{q}}dt\Big)||\nabla u||_{L^p(\Omega)}.$$

Divide now throughout by $\mathrm{Vol}(B) = \omega_n \sigma^n$

$$\Big|\fint_B u(z)d\mathbf{z} - u(x)\Big| \le \sigma^{1+\frac{n}{q}-n}\omega_n^{\frac{1}{q}-1}\Big(\int_0^1 t^{-n(1-\frac{1}{q})}dt\Big)||\nabla u||_{L^p(\Omega)}$$

$$= \sigma^{1-\frac{n}{p}}\omega_n^{-\frac{1}{p}}\Big(\int_0^1 t^{-n(\frac{1}{p})}dt\Big)||\nabla u||_{L^p(\Omega)}$$

and the integral evaluates to $\left[\frac{t^{-\frac{n}{p}+1}}{1-\frac{n}{p}}\right]\Bigg|_0^1$ which is finite iff $p > n$. We thus conclude

$$\Big| \fint_B u(z)d\mathbf{z} - u(x)\Big| \leq c(n,p)\cdot \sigma^{1-\frac{n}{p}}||\nabla u||_{L^p(\Omega)}.$$

We repeat the above computation with x replaced by y and use the triangle inequality, which gives us

$$\Big|u(x)-u(y)\Big| \leq 2c(n,p)\cdot |x-y|^{1-\frac{n}{p}}||\nabla u||_{L^p(\Omega)}$$

and subsequently

$$\frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq 2c(n,p)\cdot ||\nabla u||_{L^p(\Omega)}.$$

And concluding

$$||u||_{C^\alpha(\bar{\Omega})} = ||u||_{L^\infty(\Omega)} + \sup_{x\neq y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq C(n,p,\Omega)\cdot ||\nabla u||_{L^p(\Omega)}.$$

since both $\mathcal{C}^0$ and L^∞ norms coincide, being just $\sup_\Omega$, and finally because by our above computations we can also bound the L^∞ norm in terms of the L^p norm of Du

$$|u(x)| \leq 2c(n,p,\mathrm{diam}(\Omega))\cdot ||\nabla u||_{L^p(\Omega)}$$

so $||u||_{L^\infty(\Omega)}$ is bounded by the same RHS . ■

Compactness Theorems

Lemma. *Let Ω be a bounded domain in $\mathbb{R}^n$, and $1 \leq p < \infty$. Let S be a bounded set in $L^p(\Omega)$. In other words,*

$$\forall\, u \in S, \quad ||u||_{L^p(\Omega)} \leq M_S.$$

Suppose $\forall\, \epsilon > 0, \quad \exists\ \delta > 0$ *such that*

$$\forall\ \ u \in S,\ \forall\ \ |z| < \delta \qquad \int_{\Omega} |u(y+z) - u(y)|^p d\mathbf{y}\ <\ \epsilon.$$

Then S *is precompact in* $L^p(\Omega)$ *(denoted* $S \Subset L^p(\Omega)$*), i.e every sequence of functions in* S *has convergent subsequence ("subconverges"), or equivalently* $\bar{S}$ *is compact.*

This is an Arzelà-Ascoli type theorem: bounded equicontinuous family is precompact. We just have to show somehow that the integral equicontinuity-type condition implies equicontinuity.

Proof. Mollify u as done previously in the course

$$u_h = \int_{\mathbb{R}^n} \rho_h(x-y)u(y)d\mathbf{y}, \quad \rho_h(z) = \frac{1}{h^n}\rho(\frac{|z|}{h}).$$

Set $S_h := \{u_h, u \in S\}$.

We compute

$$\begin{aligned} u_h = \int_{\mathbb{R}^n} \rho_h(x-y)u(y)d\mathbf{y} &= \int_{\mathbb{R}^n} \rho_h(x-y)|u(y)|d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \rho_h^{\frac{1}{q}} \rho_h^{\frac{1}{p}} |u(y)|d\mathbf{y} \\ &\leq \{ \int_{\mathbb{R}^n} \rho_h \}^{\frac{1}{q}} \cdot \{\rho_h |u(y)|^{\frac{1}{p}} d\mathbf{y} \\ &\leq ||u||_{L^p(\Omega)}. \end{aligned}$$

Now

$$\begin{aligned} u_h(x+z) - u_h(x) &= \int_{\mathbb{R}^n} \big[\rho_h(x+z-y) - \rho_h(x-y)\big]u(y)d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \big[\rho_h(x-y)\big[u(y+z) - u(y)\big]d\mathbf{y} \end{aligned}$$

and the same estimate as above yields

$$u_h(x+z) - u_h(x) \leq 1 \cdot \{ \int_{\Omega} |u(y+z) - u(y)|^p \}^{\frac{1}{p}} \leq \epsilon^{\frac{1}{p}}.$$

Now by our assumption for $\delta > 0$ small enough and $|z| < \delta$ we will attain any desired ϵ on the RHS. Note $\int_{\mathbb{R}^n} \rho_h = 1$ is fixed for all h by our choice of ρ. Hence by definition we see that S_h is an equicontinuous family, and bounded WRT the $L^p(\Omega)$ norm as inside S, hence by the Arzelà-Ascoli theorem S_h is precompact *in the space* $L^p(\Omega)$.

Now $\lim_{h\to 0} S_h \to S$ as we have seen in previous lectures. So as the above estimates are independent of h, S is precompact itself in $L^p(\Omega)$. ■

Theorem **(Kodrachov)** *Let Ω be bounded in $\mathbb{R}^n$.*

$$(I) \quad p < n: \qquad W_0^{1,p}(\Omega) \subseteq L^q(\Omega) \qquad \forall \ 1 \le q < \frac{np}{n-p}.$$

$$(II) \quad p > n: \qquad W_0^{1,p}(\Omega) \subseteq \mathcal{C}^{0,\alpha}(\bar{\Omega}) \qquad \forall \ 0 < \alpha < 1 - \frac{n}{p}.$$

and moreover $W_0^{1,p}(\Omega)$ is compactly embedded in each of the RHS*s.*

We have then a curious situation– $W_0^{1,p}(\Omega) \subseteq L^{\frac{np}{n-p}} \subseteq L^q$ for $1 \le q < \frac{np}{n-p}$ but the first inclusion is only continuous! Only for q stricly smaller than $\frac{np}{n-p}$ is it compact... And similarly for the case $p > n$.

For the sake of clarity: we say $B_1 \subseteq B_2$ is *compactly embedded* if for every bounded set S in B_1, $i(S) \subseteq B_2$ is precompact, where $i : B_1 \to B_2$ is the inclusion map.

Proof. **Case** $q = 1$. By the density argument we mentioned repeatedly we assume WLOG $S \subseteq \mathcal{C}_0^1(\Omega)$ and that $M_S = 1$. Let $u \in S$. Then $||u||_{L^p(\Omega)} \le 1$, $||\mathrm{D}u||_{L^p(\Omega)} \le 1$. Hence $||u||_{L^1(\Omega)} = \int_\Omega |u(x)| \le \{\int_\Omega 1\}^{\frac{1}{q}} \{\int_\Omega |u|^p\}^{\frac{1}{p}} \le \mathrm{Vol}(\Omega)^{\frac{1}{q}} \cdot 1$, in other words S is also bounded in L^1. Once we show the condition of the Lemma holds then we will have precompactness in $L^1(\Omega)$. And indeed

$$u(y+z) - u(y) = \int_0^1 \frac{du}{dt}(y+tz)dt = \int_0^1 \nabla u(y+tz) \cdot z dt \quad \Rightarrow$$

$$\int_\Omega |u(y+z) - u(y)| d\mathbf{y} \le |z| \mathrm{Vol}(\Omega)^{\frac{1}{q}} ||\nabla u||_{L^p(\Omega)} \le c|z|.$$

Case $1 < q < \frac{np}{n-p}$. We try to find some estimates for the $L^q(\Omega)$ norm using the indispensible Hölder Inequality. Naturally we will be able to take care of boundedness of all such q together if we allude to the fact that the $\Lambda^{\frac{np}{n-p}}(\Omega)$ is bounded, indeed the L^p norms are increasing in p– first choose λ such that $q\lambda + q(1-\lambda)\dfrac{n-p}{np} = 1$

$$\{\int |u|^q\} = \{\int |u|^{q\lambda} \cdot |u|^{q(1-\lambda)}\} \leq \left\{\int \left(|u|^{q\lambda}\right)^{\frac{1}{q\lambda}}\right\}^{q\lambda} \cdot \left\{\int \left(|u|^{q(1-\lambda)}\right)^{\frac{np}{n-p}\frac{1}{q(\lambda-1)}}\right\}^{q(1-\lambda)(\frac{n-p}{np})} \quad \Rightarrow$$

$$\begin{aligned}
||u||_{L^q(\Omega)} &\leq ||u||^{\lambda}_{L^1(\Omega)} \cdot ||u||^{1-\lambda}_{L^{\frac{np}{n-p}}(\Omega)} \\
&\leq ||u||^{\lambda}_{L^1(\Omega)} \cdot c \cdot ||\nabla u||^{1-\lambda}_{L^p(\Omega)} \\
&\leq ||u||^{\lambda}_{L^1(\Omega)} \cdot c \cdot 1 \\
&\leq c(n, p, \mathrm{Vol}(\Omega)),
\end{aligned}$$

where we applied our Theorem from the previous lecture. Now note that we are done using the $q = 1$ case: S is bounded in $L^q(\Omega)$ and hence a subsequence converges in $L^q(\Omega)$, but then by the above inequality it will also converge in $L^q(\Omega)$!

Case $p > n$**.** By the Theorem of the previous lecture $W_0^{1,p}(\Omega) \subseteq \mathcal{C}^{0,\alpha}(\bar{\Omega})$ continuously. But now $\mathcal{C}^{0,\alpha}(\bar{\Omega}) \subseteq \mathcal{C}^{0,\beta}(\bar{\Omega})$ compactly for any $0 \leq \beta < \alpha$ as mentioned in one of the previous lectures. ■

Remark. Replacing $W_0^{1,p}(\Omega)$ by $W^{1,p}(\Omega)$ (the completion of $\mathcal{C}^1(\Omega)$ WRT the $W^{1,p}$ norm) in the above embedding theorems require that the domain be Lipschitz, i.e $\partial\Omega$ is of class $\mathcal{C}^{0,1}$ (this is a local requirement).