Lecture Notes for Diff Anal 2 - Larry Guth - trans. Paul Gallagher - 2/13/15

Recall the following definition of Γ:

$$\Gamma(x) = \begin{cases} c_n \frac{1}{|x|^{n-2}} & n \geq 3 \\ c_2 \log |x| & n = 2 \end{cases}$$

Note that derivatives of Γ will trivially satisfy $|\nabla \Gamma| \approx |x|^{-n+1}$ and $|\partial^2 \Gamma| \approx |x|^{-n}$.

With this notation we have already proven:

Prop 1: If $u \in C_c^4(\mathbb{R}^n)$ and $\Delta u = f$ then $u = \Gamma * f$.

Prop 2: If $f \in C_c^2(\mathbb{R}^n)$ and $u = f * \Gamma$ then

$$\partial_i \partial_j u = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} f(y) \partial_i \partial_j \Gamma(x-y) dy + \frac{1}{n} \delta_i j f(x)$$

We now aim to prove

THM 1:

$$\sum_{u \in C_c^\infty} \frac{\|\partial_i \partial_j u\|_\infty}{\|\Delta u\|_\infty} = \infty$$

THM 2 (Korn): For $0 < \alpha < 1$, $u \in C_c^2(\mathbb{R}^n)$,

$$[\partial_i \partial_j u]_\alpha \lesssim [\Delta u]_\alpha$$

Because of **Prop 1**, the general setup for proving **THM 1** is the following: Given g, we want to find an f such that

1. $\|f\|_\infty = 1$
2. $|f * g|(0) = |\int f(y) g(-y) dy|$ is very large

The best way to do this is clearly to take $f(y) = \text{sgn}(g(-y))$, so that $|f * g|(0) = \int |g(y)| dy$.

For our situation, we have that $g = \partial_i \partial_j \Gamma$, and so $\int |g| \to \infty$. Smooth out our choice of f so that it's in $C_c^\infty(\mathbb{R}^n)$. Therefore, we've proven the following:

Lemma: $\forall i, j, n, B > 0$, $\exists f_B \in C_c^\infty(B_1 \backslash \{0\})$ such that $u_B = f_B * \Gamma$, and $\|f_B\|_\infty \leq 1$ and $\partial_i \partial_j u_B(0) > B$.

Proof of THM 1: Let $w_B = u_B \eta_R$ where η_R is a cutoff function which is 1 on B_R, 0 on B_{2R}^c, and satisfies $|\partial^k \eta_R| < C_k R^{-k}$. Then $\partial_i \partial_j w_B > B$, and

$$|\Delta w_B(x)| \leq |(\Delta u) \eta_B| + 2|\nabla u_B \cdot \nabla \eta_R| + |u_B \partial^2 \eta_R|$$

But since $|u_B| \lesssim |x|^{-n+2}$ and $|\nabla u_B| \lesssim |x|^{-n+1}$, and $|(\Delta u)| \lesssim 1$, and since derivatives of η are bounded, we get that $|\Delta w_B(x)| \lesssim 1$.

Now let's work towards a proof of Korn's theorem. As setup, define

$$T_\epsilon f(x) = \int_{|x-y|>\epsilon} f(y) \partial_i \partial_j \Gamma(x-y) dy = f * K_\epsilon(x)$$

Then Korn's Theorem can be equivalently expressed as

THM 2': For $f \in C_c^\alpha$, $[T_\epsilon f]_\alpha \lesssim [f]_\alpha$

Let's take $x_1, x_2 \in \mathbb{R}^n$, $|x_1 - x_2| = d$ and normalize so $[f]_\alpha = 1$. Then we want to show that $|T_\epsilon f(x_1) - T_\epsilon f(x_2)| \lesssim d^\alpha$.

There will be three typical examples to consider, and then a general case will break down into a sum of the examples.

Ex 1: Suppose that f is supported on $B_{3d/4}(x_1) \cap B_{3d/4}(x_2)$. Then if $y \in \mathbf{supp} f$, $d \geq |x_i - y| \geq d/4$. Also, $\sup f \lesssim d^\alpha$, because $[f]_\alpha = 1$. Therefore,

$$|T_\epsilon f(x_1)| = \left| \int_{d/4<|x_1-y|<d} f(y)K(x_1-y)dy \right| \lesssim d^\alpha \int_{\text{Ann}} |x_1-y|^{-n} dy$$

Now, $|x_1 - y| \gtrsim d$ and $Vol(Ann) \lesssim d^n$, so the whole integral is less than d^α. This completes the inequality for this choice of f.

Ex 2: Now suppose that f is supported on $B_{d/2}(x_1)$. Around x_2 we can use the same argument as in **Ex 1** to get that $|T_\epsilon f(x_2)| \lesssim d^\alpha$. However, around x_1 we need to use the cancellation of the kernel, namely, that

$$0 = \int_{S_r} \partial_i \partial_j \Gamma(y) dy = \int_{S_r} K_\epsilon(y) dy$$

Using this, we have that

$$\begin{aligned}
|T_\epsilon f(x_1)| &\lesssim \int_{B_{d/2}(x_1)} \frac{|f(y) - f(x_1)|}{|x_1-y|^\alpha} K_\epsilon(x_1-y)|x_1-y|^\alpha dy \\
&\lesssim \int_{B_{d/2}(x_1)} |x_1-y|^\alpha |K_\epsilon(x_1-y)| dy \\
&\lesssim \int_{\epsilon<|x_1-y|<d/2} |x_1-y|^{-n+\alpha} \lesssim d^\alpha
\end{aligned}$$

since that last integral is actually doable.

18.156 Differential Analysis II: Partial Differential Equations and Fourier Analysis
Spring 2016