# MATH 18.152 COURSE NOTES - CLASS MEETING # 10

**18.152 Introduction to PDEs**, Fall 2011                    Professor: Jared Speck

## Class Meeting # 10: Introduction to the Wave Equation

### 1. WHAT IS THE WAVE EQUATION?

The standard wave equation for a function $u(t,x)$ (where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$) is

$$(1.0.1) \qquad -\frac{1}{c^2}\partial_t^2 u + \Delta u = 0.$$

(1.0.1) is second order and linear. The constant $c > 0$ is called the *speed* (this terminology will be justified as our course progresses), and it has dimensions of $\frac{\text{length}}{\text{time}}$. Note that heuristically speaking, if we let $c \to \infty$, then (1.0.1) becomes Laplace's equation. However, as we will see, in order to have a well-posed problem for (1.0.1), we will need to specify Cauchy (i.e. initial data) for $u$ **and also** $\partial_t u$. The fact that we need to specify Cauchy data is in stark contrast to Laplace's equation, but is analogous to the heat equation. The fact that we need to specify two pieces of Cauchy data is connected to the fact that the wave equation is second order in time.

### 2. WHERE DOES IT COME FROM?

Equation (1.0.1) arises in an incredible variety of physical contexts, especially those involving *disturbances* that propagate at a finite speed. Let's discuss how the wave equation arises as an approximation to the equations of fluid mechanics. For simplicity, let's only discuss the case of 1 spatial dimension. The equations of fluid mechanics, which are known as the *Euler equations*, take the following form in $1 + 1$ dimensions:

$$(2.0.2a) \qquad \partial_t \rho + \partial_x(\rho v) = 0,$$

$$(2.0.2b) \qquad \partial_t(\rho v) + \partial_x(\rho v^2) = -\partial_x p,$$

where $\rho(t,x)$ is the fluid *mass density*, $v(t,x)$ is the fluid *velocity*, and $p(t,x)$ is the *pressure*. Equation (2.0.2a) implies the conservation of mass, and equation (2.0.2b) is Newton's second law: the rate of change of fluid momentum is equal to the force, which is created by the pressure gradient (i.e., the $-\partial_x p$ term). The Euler equations are highly nonlinear, and we are very far from obtaining a full understanding of how their solutions behave in general.

A fundamental aspect of fluid mechanics is that the system is not closed because there are not enough equations. A common method of achieving closure is by choosing an *equation of state*, which is a relationship between the fluid variables. This relationship is often empirically determined. A commonly studied equation of state is

$$(2.0.3a) \qquad p = K\rho^\gamma$$

where $\gamma > 1$ and $K > 0$ are constants. For future use, we note that under (2.0.3a), we have

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(2.0.3b)                                     $\partial_x p = K\gamma\rho^{\gamma-1}\partial_x\rho,$

(2.0.3c)                             $\partial_x^2 p = K\gamma\rho^{\gamma-1}\partial_x^2\rho + K\gamma(\gamma-1)\rho^{\gamma-2}(\partial_x\rho)^2,$

Also for future use, we differentiate (2.0.2a) with respect to $t$ and (2.0.2b) with respect to $x$ to deduce that

(2.0.4a)                    $\partial_t^2\rho + \rho\partial_t\partial_x v + v\partial_t\partial_x\rho + \partial_t\rho\partial_x v + \partial_t v\partial_x\rho = 0,$

(2.0.4b)      $\rho\partial_t\partial_x v + v\partial_t\partial_x\rho + \partial_t\rho\partial_x v + \partial_t v\partial_x\rho + \partial_x^2\rho + 4\rho v\partial_x^2 v + 2\partial_x\rho\partial_x v = -\partial_x^2 p.$

The theory of acoustics is based on *linearizing* (i.e. throwing away the nonlinear terms) the equations (2.0.4a) - (2.0.4b) around the static solutions $\rho = \bar{\rho} = const > 0$, $v = 0$, $p = \bar{p} = const > 0$. These static solutions describe a fluid at rest. Let's assume that we make a small perturbation of this solution, i.e., that $v$ is small, and that

(2.0.5)                                          $\rho = \bar{\rho} + \delta,$

where $\delta(t, x)$ is a small function.

Using the expansion (2.0.5), we now throw away (with the help of (2.0.3c)) all of the quadratic and higher-order small terms from (2.0.4a) - (2.0.4b) to obtain the following **approximating** system (the quantities that are assumed to be small are $v$, $\delta$, and all of their partial derivatives):

(2.0.6a)                                   $\partial_t^2\delta + \bar{\rho}\partial_t\partial_x v = 0,$

(2.0.6b)                                   $\bar{\rho}\partial_t\partial_x v = -K\gamma\bar{\rho}^{\gamma-1}\partial_x^2\delta.$

Comparing (2.0.6a) and (2.0.6b), we see that $\delta$ verifies the following **approximating** equation

(2.0.7)                                $-\partial_t^2\delta + K\gamma\bar{\rho}^{\gamma-1}\partial_x^2\delta = 0.$

Equation (2.0.7) is a wave equation for the perturbation $\delta(t, x)$! It models the propagation of *sound waves*. This is the linear theory of acoustics! Note that the speed associated to the equation (2.0.7) depends on the background density $\bar{\rho}$:

(2.0.8)                                       $c = \sqrt{K\gamma\bar{\rho}^{\gamma-1}}.$

When $\gamma > 1$, higher background density $\implies$ faster sound speed propagation.

**Remark 2.0.1.** For air under "normal" atmospheric conditions, $\gamma = 1.4$ is a pretty good model.

## 3. Some Well-Posed Problems

Recall that well-posed PDEs have three important properties:

- Given suitable data, a solution exists.
- The solution is unique.
- The solution depends continuously on the data.

Perhaps the most often studied well-posed problem for the wave equation is the *global Cauchy problem* in $1 + n$ spacetime dimensions:

(3.0.9a) $$-\partial_t^2 u(t, x) + \Delta_x u(t, x) = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

(3.0.9b) $$u(0, x) = f(x), \qquad x \in \mathbb{R}^n,$$

(3.0.9c) $$\partial_t u(0, x) = g(x), \qquad x \in \mathbb{R}^n.$$

We now mention some additional well-posed problems in the case of $1 + 1$ dimensions. We assume that $u$ verifies the wave equation for $(t, x) \in (-\infty, \infty) \times [0, L]$ and that Cauchy data is given:

(3.0.10a) $$-\partial_t^2 u(t, x) + \partial_x^2 u(t, x) = 0, \qquad (t, x) \in \mathbb{R} \times [0, L],$$

(3.0.10b) $$u(0, x) = f(x), \qquad x \in [0, L],$$

(3.0.10c) $$\partial_t u(0, x) = g(x), \qquad x \in [0, L].$$

Unlike in the case of (3.0.9a) - (3.0.9c), because of the finiteness of the interval $[0, L]$, we need to supplement (3.0.10a) - (3.0.10c) with additional conditions in order to generated a well-posed problem. Here are some well-known ways of generating a well-posed problem; they are essentially the same as in the case of the heat equation.

(1) *Dirichlet data*: also specifying $u(t, 0) = a(t)$, $u(t, L) = b(t)$ for $t > 0$
(2) *Neumann data*: also specifying $\partial_x u(t, 0) = a(t)$, $\partial_x u(t, L) = b(t)$ for $t > 0$
(3) *Robin data*: also specifying $\partial_x u(t, 0) = ku(t, 0)$, $\partial_x u(t, L) = -ku(t, L)$ for $t > 0$, where $k > 0$ is a constant
(4) *Mixed data*: e.g. one kind of data at $x = 0$, and a different kind at $x = L$

## 4. $1 + 1$ SPACETIME DIMENSIONS

Let's consider the wave equation with speed $c$ in $1 + 1$ dimensions:

(4.0.11) $$-c^{-2} \partial_\tau^2 u(\tau, x) + \partial_x^2 u(\tau, x) = 0.$$

Let's first note the following fact: if $f, g$ are any differentiable functions, then $u(x, \tau) \overset{\text{def}}{=} f(x - c\tau)$ and $u(x, \tau) \overset{\text{def}}{=} g(x + c\tau)$ solve (4.0.11). The first is called a *right-traveling wave*, and the second is called a *left-traveling* wave. To visualized wave propagation in $1 + 1$ dimensions, you can imagine that the graph of $f(\cdot)$ and $g(\cdot)$ are translated to the right/left at a speed $c$. This gives a good idea of what wave motion looks like in $1 + 1$ dimensions. In particular, the amplitudes of the traveling wave solutions are preserved in time. As we will see, wave propagation in higher dimensions is quite different. In higher dimensions, the amplitudes decay in time due to the spreading out of the waves. You will study the case of $1 + 3$ spatial dimension in one of your homework exercises; you will show that in this case, the amplitudes decay at a rate of order $t^{-1}$ as $t \to \infty$.

**Remark 4.0.2.** Not all wave solutions in $1 + 1$ dimensions are traveling waves; see Theorem 4.1.

By making the change of variables $t \overset{\text{def}}{=} c\tau$, we can transform equation (4.0.11) into a wave equation with speed equal to $1$:

(4.0.12) $$-\partial_t^2 u(t, x) + \partial_x^2 u(t, x) = 0.$$

This makes our life a bit easier. Let's now consider the global Cauchy problem by supplementing (4.0.12) with the initial data

(4.0.13)                     $u(0,x) = f(x),$                                    $\partial_t u(0,x) = g(x).$

As we will see, (4.0.12) + (4.0.13) has a unique solution that has a nice representation.

**Theorem 4.1 (d'Alembert's formula).** *Assume that* $f \in C^2(\mathbb{R})$ *and* $g \in C^1(\mathbb{R})$. *Then the unique solution* $u(t,x)$ *to* (4.0.12) + (4.0.13) *satisfies* $u \in C^2([0,\infty) \times \mathbb{R})$ *and can be represented by* ***d'Alembert's formula****:*

$$(4.0.14) \qquad u(t,x) = \frac{1}{2}\Big(f(x+t) + f(x-t)\Big) + \frac{1}{2}\int_{z=x-t}^{z=x+t} g(z)\, dz.$$

**Remark 4.0.3.** For the wave equation $-c^{-2}\partial_t^2 u + \partial_x^2 u = 0$ formula (4.0.14) is replaced with

$$(4.0.15) \qquad u(t,x) = \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big) + \frac{1}{2c}\int_{z=x-ct}^{z=x+ct} g(z)\, dz.$$

**Remark 4.0.4.** Equation (4.0.14) illustrates the *finite speed of propagation* property associated to the wave equation. More precisely, the value of the solution at $(t,x)$ is only influenced by the "initial data interval" $\{(0,y) \mid x - t \le y \le x + t\}$; changes to the initial data (4.0.13) outside of this interval have no effect on the solution at $(t,x)$. We will reexamine this property later in the course with the help of energy methods.

*Proof.* To derive (4.0.14), it is convenient to introduce a change of variables called *null coordinates*:

$$(4.0.16) \qquad\qquad\qquad\qquad q \stackrel{\text{def}}{=} t - x,$$

$$(4.0.17) \qquad\qquad\qquad\qquad s \stackrel{\text{def}}{=} t + x.$$

The chain rule implies the following relationships between partial derivatives:

$$(4.0.18) \qquad\qquad \partial_q = \frac{1}{2}(\partial_t - \partial_x), \qquad\qquad\qquad \partial_s = \frac{1}{2}(\partial_t + \partial_x),$$

$$(4.0.19) \qquad\qquad \partial_t = \partial_q + \partial_s, \qquad\qquad\qquad\qquad \partial_x = \partial_s - \partial_q.$$

The operators $\partial_q$ and $\partial_s$ can be viewed as directional derivatives in the $(t,x)$ Cartesian spacetime direction $.5(1,-1)$ and $.5(1,1)$ respectively. These *null directions*, which are sometimes called *characteristic directions*, are extremely important. In the future, we will discuss the notion of a characteristic direction in a general setting.

It is now easy to see that (4.0.12) takes the following form in null coordinates:

$$(4.0.20) \qquad\qquad\qquad\qquad \partial_s \partial_q u = 0.$$

Integrating (4.0.20) with respect to $s$, we have that

$$(4.0.21) \qquad\qquad\qquad\qquad \partial_q u = H(q),$$

where $H$ is a function of $q$.

Note that the value of $q$ is the same for the pair of Cartesian spacetime points $(\tau, y)$ and $(0, y - \tau)$. Thus, using the initial conditions (4.0.13), we have that

$$(4.0.22) \qquad \partial_q u(\tau, y) = \partial_q u(0, y - \tau) = \left(\frac{1}{2}(\partial_t - \partial_x)u\right)(0, y - \tau) = \frac{1}{2}\left(g(y - \tau) - f'(y - \tau)\right).$$

Similarly, interchanging the partial derivatives in (4.0.20) to deduce $\partial_s \partial_q u = 0$, we conclude that

$$(4.0.23) \qquad \partial_s u(\tau, y) = \frac{1}{2}\left(g(y + \tau) + f'(y + \tau)\right).$$

Adding (4.0.22) and (4.0.23), and using (4.0.18), we have that

$$(4.0.24) \qquad \partial_t u(t, x) = \frac{1}{2}\left(f'(x + t) - f'(x - t) + g(x + t) + g(x - t)\right).$$

Integrating (4.0.24) in time with respect to $t$ from $0$ to $t$, and again using the initial conditions (4.0.13), we have that

$$(4.0.25) \qquad u(t, x) = \overbrace{u(0, x)}^{f(x)} + \frac{1}{2}\left(f(x + t) - f(x) + f(x - t) - f(x)\right) + \frac{1}{2}\int_{\tau = 0}^{t} g(x + \tau) + g(x - \tau)\, d\tau$$
$$= \frac{1}{2}\left(f(x + t) + f(x - t)\right) + \frac{1}{2}\int_{z = x - t}^{z = x + t} g(z)\, dz,$$

where to derive the last equality, we made the integration change of variables $z = x + \tau$ for the $g(x + \tau)$ term, and the change of variables $z = x - \tau$ for the $g(x - \tau)$ term. We have thus derived (4.0.14).

$\square$

Without a lot of additional effort, we can extend Theorem 4.1 to apply to the following initial + boundary value PDE in $1 + 1$ dimensions; the result is stated and proved in the next corollary. This PDE would arise in the study of e.g. the following idealized problem: a description of the propagation of waves on an infinitely long vibrating string with one end fixed. Furthermore, the corollary will later play a role in our extension of Theorem 4.1 to the case of $1 + 3$ dimensions.

**Corollary 4.0.1.** *Let $f \in C^2([0, \infty))$, $g \in C^1([0, \infty))$, and assume that $f(0) = g(0) = 0$. Then the unique solution to the following $1 + 1$ dimensional initial + boundary value problem*

$$(4.0.26a) \qquad -\partial_t^2 u(t, x) + \partial_x^2 u(t, x) = 0, \qquad (t, x) \in [0, \infty) \times (0, \infty),$$
$$(4.0.26b) \qquad u(t, 0) = 0, \qquad t \in [0, \infty),$$
$$(4.0.26c) \qquad u(0, x) = f(x), \qquad x \in (0, \infty),$$
$$(4.0.26d) \qquad \partial_t u(0, x) = g(x), \qquad x \in (0, \infty)$$

*satisfies $u \in C^2([0, \infty) \times [0, \infty))$. Furthermore, it can be represented as*

$$(4.0.27) \qquad u(t, x) = \begin{cases} \frac{1}{2}\left(f(x + t) + f(x - t)\right) + \frac{1}{2}\int_{z = |x - t|}^{z = x + t} g(z)\, dz, & \text{if } 0 \le t \le x, \\ \frac{1}{2}\left(f(x + t) - f(t - x)\right) + \frac{1}{2}\int_{z = |x - t|}^{z = x + t} g(z)\, dz, & \text{if } 0 \le x \le t. \end{cases}$$

*Proof.* The idea is that if we extend $u$ to be odd in $x$, then we can reduce the problem to the case of Theorem 4.1. Motivated by this, we define

$$(4.0.28) \qquad \widetilde{u}(t,x) \overset{\text{def}}{=} \begin{cases} u(t,x), & \text{if } t \geq 0, x \geq 0, \\ -u(t,-x), & \text{if } t \geq 0, x \leq 0, \end{cases},$$

$$(4.0.29) \qquad \widetilde{f}(x) \overset{\text{def}}{=} \begin{cases} f(x), & \text{if } x \geq 0, \\ -f(-x), & \text{if } x \leq 0, \end{cases},$$

$$(4.0.30) \qquad \widetilde{g}(x) \overset{\text{def}}{=} \begin{cases} g(x), & \text{if } x \geq 0, \\ -g(-x), & \text{if } x \leq 0. \end{cases}$$

Since $u(t,x)$ solves (4.0.26a), it follows that $\widetilde{u}(t,x)$ is a solution to the wave equation (4.0.12) for $(t,x) \in \mathbb{R} \times \mathbb{R}$ with initial data $\widetilde{u}(0,x) = \widetilde{f}(x)$, $\partial_t \widetilde{u}(t,x) = \widetilde{g}(x)$. Thus, by (4.0.14), we have that

$$(4.0.31) \qquad \widetilde{u}(t,x) = \frac{1}{2}\left(\widetilde{f}(x+t) + \widetilde{f}(x-t)\right) + \frac{1}{2}\int_{z=x-t}^{z=x+t} \widetilde{g}(z)\,dz.$$

The expression (4.0.27) now easily follows from considering (4.0.31) separately in the spacetime regions $\{(t,x) \,|\, 0 \leq t \leq x\}$ and $\{(t,x) \,|\, 0 \leq x \leq t\}$, and from the definitions (4.0.28) - (4.0.30); note that in the case $\{(t,x) \,|\, 0 \leq t \leq x\}$, since $\widetilde{g}$ is odd, the part of the integral from $x-t$ to $t-x$ cancels and thus the only net contribution comes from the integration interval $[|x-t|, x+t]$.

$\square$

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