MATH 18.152 - PROBLEM SET 5

18.152 Introduction to PDEs, Fall 2011 — Professor: Jared Speck

Problem Set 5, **Due: at the start of class on 10-13-11**

I. Let $B_1(0)$ denote the solid open unit ball in $\mathbb{R}^3$, and let $\partial B_1(0)$ denote its boundary. Recall that the Green function $G(x,y)$ for $B_1(0)$ satisfies

$$G(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi}\frac{1}{|x|\big|\frac{1}{|x|^2}x - y\big|}, \qquad x,y \in B_1(0), \qquad x \neq 0, \tag{0.0.1a}$$

$$G(0,y) = -\frac{1}{4\pi|y|} + \frac{1}{4\pi}, \qquad y \in B_1(0), \tag{0.0.1b}$$

$$\nabla_{\hat{N}(\sigma)}G(x,\sigma) = \frac{1-|x|^2}{4\pi}\frac{1}{|x-\sigma|^3}, \qquad x \in B_1(0), \qquad \sigma \in \partial B_1(0). \tag{0.0.1c}$$

Using (0.0.1a), directly show that $G(x,y) \leq 0$ holds for all $x,y \in B_1(0)$.

II. Let $B_1(0)$ denote the solid open unit ball in $\mathbb{R}^3$, and let $\partial B_1(0)$ denote its boundary. Let $f(x)$ be smooth (i.e., infinitely differentiable) function on $B_1(0)$, let $g(\sigma)$ be a smooth function on $\partial B_1(0)$, and let $u(x)$ be the unique smooth solution to

$$\begin{aligned} \Delta u(x) &= f(x), && x \in B_1(0), \\ u(\sigma) &= g(\sigma), && \sigma \in \partial B_1(0). \end{aligned}$$

Recall that the solution $u(x)$ can be represented as

$$u(x) = \int_\Omega f(y)G(x,y)\,d^3y + \int_{\partial\Omega} g(\sigma)\nabla_{\hat{N}}G(x,\sigma)\,d\sigma. \tag{0.0.2}$$

First show that $\int_\Omega G(x,y)\,d^3y = \frac{1}{6}\big[(x^1)^2 + (x^2)^2 + (x^3)^2\big] - \frac{1}{6}$, where $x = (x^1,x^2,x^3)$.

Then use Problem **I** to conclude that $\int_\Omega |G(x,y)|\,d^3y \leq \frac{1}{6}$ whenever $x \in B_1(0)$.

Hint: Try setting $u(x^1,x^2,x^3) = \frac{1}{6}\big[(x^1)^2 + (x^2)^2 + (x^3)^2\big] - \frac{1}{6}$.

Finally, show that there exists a constant $C > 0$ which does *not* depend on f or g such that

$$\max_{x\in B_1(0)}|u(x)| \leq C\big(\max_{x\in B_1(0)}|f(x)| + \max_{\sigma\in\partial B_1(0)}|g(\sigma)|\big).$$

III. Let $u(x)$ be a harmonic function on $\mathbb{R}^3$, and assume that $|u(x)| \leq \ln(|x|+1)$ holds for all $x \in \mathbb{R}^3$. Show that $u(x) = 0$ for all x.

Hint: For each fixed $R > 0$, consider the function $v(x) \overset{\text{def}}{=} u(x) + \ln(R+1)$. Since v is non-negative, for each fixed $|x| \leq R$, you can apply Harnack's inequality to $v(x)$ on the domain $B_R(0)$. Then you can allow $R \to \infty$ in Harnack's inequality, and with the help of L'Hôpital's rule, you should be able to reach the desired conclusion.

Remark 0.0.1. Roughly speaking, this problem shows that the non-constant harmonic functions are not allowed to grow at merely the very slow rate $\ln(|x| + 1)$ as $|x| \to \infty$.

IV. Problem **3.14** on pg. 153.

Hint: In addition to the hints for part a) given in the book (which are leading you towards the comparison principle for harmonic functions), also use the fact that $G(x, y) \to -\infty$ as $y \to x$, for each fixed $x \in \Omega$. This fact follows from the expression $G(x, y) = \Phi(x-y) - \phi(x, y)$, from the fact that $|\phi(x, y)|$ is uniformly bounded for y near x, and from the fact that $\Phi(x - y) \to -\infty$ as $y \to x$; this last fact follows from the expression for $\Phi(x - y)$ given in class.

For part b), use the following strategy, which differs from the one in your book: for each pair of points $a, b \in \Omega$, define $u(x) = G(x, a)$ and $v(x) = G(x, b)$. Apply Green's identity to u, v on the domain $\Omega_\epsilon \overset{\text{def}}{=} \Omega \backslash (B_\epsilon(a) \cup B_\epsilon(b))$. Notice that if ϵ is small, then u, v are harmonic on Ω_ϵ and that $\partial\Omega_\epsilon = \partial\Omega \cup -\partial B_\epsilon(a) \cup -\partial B_\epsilon(b)$. Then let $\epsilon \downarrow 0$. In applying Green's identity, you will have to handle 3 surface integrals: over $\partial\Omega$, $\partial B_\epsilon(a)$, and $\partial B_\epsilon(b)$. The first surface integral should be 0, the second should converge to $v(a)$ as $\epsilon \downarrow 0$, and the third should converge to $-u(b)$. The final conclusion should be that $v(a) = u(b)$. You will also need to use some of the calculations that we did for the proof of the "**Representation formula for** u" Proposition in class, as well as the fact that $|\phi(x, y)|$ and $|\nabla\phi(x, y)|$ are uniformly bounded for y near x.

Remark 0.0.2. Note that for some of the quantities we defined in class (such as the fundamental solution), we have chosen the opposite sign convention compared to the one in your book. For the sake of consistency, use the sign conventions we introduced in class. In particular, for part a), you should prove that $G(x, y) < 0$.

V. Problem **3.21** on pg. 155. **Hint:** For part a), use the fact that for a spherically symmetric function ϕ defined on $\mathbb{R}^3$, $\Delta\phi = \partial_r^2\phi + \frac{2}{r}\phi$. Then the problem can be reduced to solving a simple ODE for the quantity $r\phi$.

For part b), revisit the proof of the theorem from Class Meeting # 7 in which we solved Poisson's equation in $\mathbb{R}^n$.

Remark 0.0.3. In part a), you are deriving a fundamental solution for the Helmholtz equation, up to a constant factor c. In part b), you are finding the correct constant c.

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