MATH 18.152 - PROBLEM SET 11

18.152 Introduction to PDEs, Fall 2011	Professor: Jared Speck
Problem Set 11, **Due: at the start of class on 12-8-11**	

I. Let $N(q)$ be an invertible $(1+n)\times(1+n)$ matrix with entries $N_{\mu\nu}(q)$ that are functions of a parameter q. We denote the entries of the inverse of $N(q)$ by $(N^{-1})^{\mu\nu}(q)$. Show that

$$\frac{d}{dq}(N^{-1})^{\mu\nu} = -(N^{-1})^{\mu\alpha}(N^{-1})^{\beta\nu}\frac{d}{dq}N_{\alpha\beta}. \tag{0.0.1}$$

Hint: Try differentiating the relation $N^{-1}(q)N(q) = I$ with respect to q, where $I \overset{\text{def}}{=} \text{diag}(1, 1, \cdots, 1)$ is the $(1+n)\times(1+n)$ identity matrix.

II. Let A, B be $(1+n)\times(1+n)$ matrices (with e.g. entries A_ν^μ, $0 \leq \mu, \nu \leq n$), and let $\|A\|$ be the matrix norm defined by

$$\|A\| \overset{\text{def}}{=} \sqrt{\sum_{0\leq\mu,\nu\leq n} |A_\nu^\mu|^2}. \tag{0.0.2}$$

a) Show that $\|AB\| \leq \|A\|\|B\|$. Here, AB is the standard matrix product, which is defined by

$$(AB)_\nu^\mu \overset{\text{def}}{=} A_\alpha^\mu B_\nu^\alpha \qquad \text{(Einstein summation convention for } \alpha\text{)}. \tag{0.0.3}$$

Hint: Note that $\|AB\|^2 = A_\alpha^\mu A_\beta^\nu B_\nu^\alpha B_\mu^\beta$ and think about the Cauchy-Schwarz inequality.

b) Let $I \overset{\text{def}}{=} \text{diag}(1, 1, \cdots, 1)$ denote the $(1+n)\times(1+n)$ identity matrix. Show that there exists a number $\delta > 0$ such that if $\|A\| \leq \delta$, then

$$\det(I + A) = I + A_\alpha^\alpha + \|A\|^2\mathcal{R}(A), \tag{0.0.4}$$

where $\mathcal{R}$ is a bounded function of A for $0 < \|A\| \leq \delta$. Above, $A_\alpha^\alpha \overset{\text{def}}{=} \sum_{0\leq\alpha\leq n} A_\alpha^\alpha$ is the trace of A.

c) Show that if N is any positive integer, then

$$I - A^{N+1} = (I - A)(I + A + A^2 + \cdots + A^N). \tag{0.0.5}$$

d) Use parts **a)**, **b)**, and **c)** to show that there exists a number $\gamma > 0$ such that if $\|A\| \leq \gamma$, then $(I - A)$ is an invertible matrix and $(I - A)^{-1}$ can be expanded in a convergent matrix series as follows:

$$(I-A)^{-1} = I + \sum_{N=1}^{\infty} A^N. \tag{0.0.6}$$

Notation: In the next two problems, we use the partial derivative notation $\nabla_\mu \overset{\text{def}}{=} \frac{\partial}{\partial x^\mu}$, where $x = (x^0, x^1, \cdots, x^n)$ denotes standard coordinates on $\mathbb{R}^{1+n}$.

III. Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function, and consider the Lagrangian

$$\mathcal{L} \overset{\text{def}}{=} -\frac{1}{2}(m^{-1})^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi). \tag{0.0.7}$$

Above, $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ is a function on the spacetime $\mathbb{R}^{1+n}$ and $m^{-1} = \mathrm{diag}(-1, 1, 1, \cdots, 1)$ is the standard Minkowski metric.

a) Show that (0.0.7) is a coordinate invariant Lagrangian.

b) Show that the Euler-Lagrange equation corresponding to the Lagrangian (0.0.7) is

$$\square_m \phi = V'(\phi), \tag{0.0.8}$$

where $\square_m \overset{\text{def}}{=} (m^{-1})^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the standard Minkowski wave operator.

c) Compute the energy-momentum tensor $T^{\mu\nu}$ corresponding to the Lagrangian (0.0.7).

d) Directly show that

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nu = 0, 1, \cdots, n \tag{0.0.9}$$

whenever ϕ is a C^2 solution to (0.0.8).

Remark 0.0.1. The special case $V(\phi) = \frac{\kappa^2}{2}\phi^2$ leads to the *Klein-Gordon equation*. The parameter $\kappa \geq 0$ is known as the *mass*, so that the standard linear wave equation is the "massless" Klein-Gordon equation.

IV. Let $x = (x^0, \cdots, x^n)$ denote standard coordinates on $\mathbb{R}^{1+n}$. Let $Y : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be a smooth vectorfield. For each sufficiently small $|\epsilon|$ consider the change of coordinates $\widetilde{x}$ implicitly defined by solving the following ODE initial value problem:

$$\frac{d}{d\epsilon}\widetilde{x}^\mu(\epsilon) = Y^\mu(\widetilde{x}), \tag{0.0.10}$$

$$\widetilde{x}^\mu(\epsilon = 0) = x^\mu. \tag{0.0.11}$$

Assume that the inverse Minkowski metric $(m^{-1})^{\mu\nu} = \mathrm{diag}(-1, 1, 1, \cdots, 1)$ is invariant under the change of coordinates $x \to \widetilde{x}$. More precisely, assume that for all small $|\epsilon|$, we have that

$$(\widetilde{m}^{-1})^{\mu\nu} = (m^{-1})^{\mu\nu} = \mathrm{diag}(-1, 1, 1, \cdots, 1), \tag{0.0.12}$$

where $(\widetilde{m}^{-1})^{\mu\nu} \overset{\text{def}}{=} M_\alpha^\mu M_\beta^\nu (m^{-1})^{\alpha\beta}$, and $M_\nu^\mu \overset{\text{def}}{=} \frac{\partial \widetilde{x}^\mu}{\partial x^\nu}$. The vectorfield Y is said to generate a "continuous symmetry" of m^{-1}.

a) Differentiate the relation (0.0.12) with respect to ϵ and set $\epsilon = 0$ to deduce that

$$(m^{-1})^{\alpha\nu}\nabla_\alpha Y^\mu + (m^{-1})^{\mu\alpha}\nabla_\alpha Y^\nu = 0. \tag{0.0.13}$$

b)Let $\mathfrak{K}$ be a compact subset of spacetime with a smooth boundary, and let ϕ be a C^2 solution to (0.0.8). Let Y be a smooth vectorfield, and assume that (0.0.13) holds for $x \in \mathfrak{K}$. Revisit the proof of the theorem from class in which we derived the divergence-free property of the energy-momentum tensor. In particular, recall that in the proof, we showed that

$$\int_{\mathfrak{K}} T^{\mu\nu}\left\{m_{\alpha\nu}\nabla_\mu Z^\alpha + m_{\mu\alpha}\nabla_\nu Z^\alpha\right\} d^{1+n}x = 0 \tag{0.0.14}$$

holds for any smooth vectorfield Z, where $T^{\mu\nu}$ is the energy-momentum tensor corresponding to the Lagrangian (0.0.7). Using (0.0.14) with $Z = Y$, show that

$$\int_{\mathfrak{K}} \nabla_\mu\big(T^{\mu\nu}Y_\nu\big)\, d^{1+n}x = 0. \tag{0.0.15}$$

c) Explain how the identity (0.0.15) can be used to derive conserved quantities for C^2 solutions to the Euler-Lagrange equation (0.0.8).

d) Let $\phi(t, x^1, \cdots, x^n)$ be a C^2 solution to (0.0.7) (note that we are using the alternate notation $t \stackrel{\text{def}}{=} x^0$). For simplicity, assume that $\phi(t,\cdot)$ is compactly supported for each fixed t. Using (0.0.15) in the case $Y^\mu = (-1, 0, 0, \cdots, 0)$, prove that the conservation law

$$\begin{aligned}(0.0.16)\qquad &\int_{\mathbb{R}^n} \frac{1}{2}|\nabla\phi(t, x^1, x^2, \cdots, x^n)|^2 + V\big(\phi(t, x^1, x^2, \cdots, x^n)\big)\, d^n x \\ &= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla\phi(t, x^1, x^2, \cdots, x^n)|^2 + V\big(\phi(0, x^1, x^2, \cdots, x^n)\big)\, d^n x\end{aligned}$$

holds for any $t \geq 0$. Above, $|\nabla\phi|^2 \stackrel{\text{def}}{=} (\nabla_t\phi)^2 + \sum_{j=1}^n (\nabla_j\phi)^2$ is the Euclidean norm of the spacetime gradient of ϕ, and $d^n x \stackrel{\text{def}}{=} dx^1 dx^2 \cdots dx^n$ denotes integration over the spatial variables $(x^1, x^2, \cdots, x^n)$.

Remark 0.0.2. This problem shows that "continuous symmetries" (generated by a flow vectorfield Y) of the Lagrangian generate conserved quantities for solutions to the corresponding Euler-Lagrange equation. The principle that a symmetry of a differential equation leads to a conservation law is widely applicable. The original ideas go back to Emmy Noether, who published a theorem (known as Noether's theorem) in 1918 containing the first general application of these methods. The modern point of view, which was established earlier in the course, is to study the properties of the vectorfields of the form $J^\mu \stackrel{\text{def}}{=} T^{\mu\nu}X_\nu$ for a larger class of X than the ones that correspond to exact symmetries. This modern point of view is therefore a descendant of Noether's theorem.