PDE Examples Sheet 2

Problem 1. Let u be a function on the ball $B_1(0) \subset \mathbb{R}^2$ with $\int_{B_1(0)} |\nabla u|^p <]infty$ for some constant $p > 2$. Show that u is holder continuous. [Hint: Use Morrey on $\int 1.|\nabla u|^2]$

Problem 2. Let $u : \mathbb{R}^n \to \mathbb{R}$, and define $\text{osc}_{B_r(x)} u = \sup_{B_r(x)} u - \inf_{B_r(x)} u$. Show that if there is some constant $0 < \gamma < 1$ with

$$\text{osc}_{B_r(x)} u \leq \gamma \text{osc}_{B_{2r}(x)} u$$

for all x and all r then u is Holder continuous.

Problem 3. Let L be a uniformly elliptic 2nd order operator in divergence form taking

$$Lf = \frac{\partial}{\partial x_i} A_{ij} \frac{\partial f}{\partial x_j}.$$

Let u be a function with $Lu \geq 0$, and $\Phi : \mathbb{R} \to \mathbb{R}$ a function with $\Phi', \Phi'' \geq 0$. Show that $L(\Phi(u)) \geq 0$.

Problem 4. Let L be an operator as in question 3, and let u be an L harmonic function. Prove that $|\nabla u|^2$ is holder contiuous. [This is likely to be difficult.]