MEASURE AND INTEGRATION: LECTURE 23

Lebesgue's differentiation theorem.

Theorem 0.1. *Let $f \in L^1(\mathbb{R}^n)$. Then for almost every $x \in \mathbb{R}^n$,*

$$\lim_{r\to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \; dy = 0.$$

In particular, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y) \; dy = f(x).$$

Remark. This looks like the FTOC I almost everywhere: the derivative of the integral of $f = f$. Next time, prove this and show it implies FTOC in the case of $\mathbb{R}$.

Proof of theorem. Obviously,

$$\begin{aligned}
&\left| \left(\frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y) \; dy \right) - f(x) \right| \\
&= \left| \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} (f(y) - f(x)) \right| \\
&\le \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \; dy.
\end{aligned}$$

Thus, the particular case of the theorem follows from the first statement.

Recall that if $f \in L^1$, we can define the maximal function Mf, and

$$\lambda(\{x \mid Mf(x) \ge t\}) < \frac{3^n \|f\|_1}{t}.$$

Also,

$$\{x \mid Mf(x) \ge t\} = \bigcap_{j=1}^{\infty} \{x \mid Mf(x) > t - 1/j\},$$

so

$$\lambda(\{x \mid Mf(x) \ge t\} \le \frac{3^n \|f\|_1}{t}.$$

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Define

$$f^*(x) = \limsup_{r\to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \; dy.$$

We want to show that $f^*(x) = 0$ a.e. The function f^* has the following properties.

(1) $f^* \geq 0$.
(2) $(f+g)^* \leq f^* + g^*$.

Proof.

$$\begin{aligned}
&\int_{B(x,r)} |f(y) + g(y) - f(x) - g(x)| \; dy \\
&= \int_{B(x,r)} |f(y) - f(x) + g(y) - g(x)| \; dy \\
&\leq \int_{B(x,r)} |f(y) - f(x)| \; dy + \int_{B(x,r)} |g(y) - g(x)| \; dy.
\end{aligned}$$

□

(3) If g is continuous at x, then $g^*(x) = 0$.

Proof. For any $\epsilon > 0$, there exists δ such that $|g(y) - g(x)| \leq \epsilon$ for all $y \in B(x,\delta)$. So, for $0 < r \leq \delta$,

$$\frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |g(y) - g(x)| \; dy \leq \frac{1}{\lambda(B(x,r))} \epsilon \lambda(B(x,r)) = \epsilon.$$

Thus, $g^*(x) < \epsilon$ for any ϵ, and hence $g^*(x) = 0$. □

Note that this is FTOC for continuous functions. The strategy is that we know it is true for continuous functions, so we will approximate $f \in L^1(\mathbb{R}^n)$ by $g \in C^0(\mathbb{R}^n)$.

(4) If g is continuous, then $(f-g)^* = f^*$.

Proof.

$$(f-g)^* \leq f^* + (-g)^* = f^*$$

and

$$f^* \leq (f-g)^* + g^* = (f-g)^*.$$

□

(5) $f^* \leq Mf + |f|$.

Proof.

$$\begin{aligned}
&\frac{1}{\lambda(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\ dy \\
&\le \frac{1}{\lambda(B(x,r))}\int_{B(x,r)} (|f(y)|+|f(x)|)\ dy \\
&= \left(\frac{1}{\lambda(B(x,r))}\int_{B(x,r)} |f(y)|\ dy\right) + |f(x)| \\
&\le Mf + |f(x)|
\end{aligned}$$

□

(6) Have not proved that f^* is measurable, but claim

$$\lambda^*(\{x \mid f^*(x) > t\}) \le \frac{2(3^n+1)}{t}\|f\|_1 \qquad \text{for all } 0 < t < \infty.$$

Proof. If $f^*(x) > t$ at x, then from (5),

$$t < Mf(x) + |f|(x),$$

and so either $Mf(x) > t/2$ or $|f(x)| > t/2$. So,

$$\{f^* > t\} \subset \{Mf > t/2\} \cup \{|f| > t/2\}.$$

Thus,

$$\begin{aligned}
\lambda^*(\{x \mid f^*(x) > t\}) &\le \lambda(\{x \mid Mf(x) > t/2\}) + \lambda(\{x \mid\ |f(x)| > t/2\}) \\
&\le \frac{3^n\|f\|_1}{t/2} + \frac{\|f\|_1}{t/2}.
\end{aligned}$$

The last step used the theorem from last time and Chebyshev's inequality for L^1 functions. □

To finish the proof, given $\epsilon > 0$, from the approximation theorem ($C_c(\mathbb{R}^n)$ dense in L^1), there exists $g \in C_c(\mathbb{R}^n)$ with $\|f-g\|_1 \le \epsilon$ ($f^* = (f-g)^*$.) Thus,

$$\begin{aligned}
\lambda^*(\{x \mid f^*(x) > t\}) &= \lambda^*(\{x \mid (f-g)^*(x) > t\}) \\
&\le \frac{2(3^n+1)}{t}\|f-g\|_1 \\
&\le \frac{2(3^n+1)}{t}\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, $\lambda^*(\{x \mid f^*(x) > t\}) = 0$. In particular, $\lambda^*(\{x \mid f^*(x) > 1/k\}) = 0$ for all k, and $\{x \mid f^*(x) > 0\} = \cup_{k=1}^{\infty}\{x \mid f^*(x) > 1/k\}$. Since countable union $\Rightarrow \lambda(\{x \mid f^*(x) > 0\}) = 0$. Since $f^* \ge 0$, $f^* = 0$ almost everywhere. □

Lebesgue set. Let $f \in L^1(\mathbb{R}^n)$. Then $x \in \mathbb{R}^n$ is in the *Lebesgue set* of f if there exists a number A such that

$$\lim_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - A| \ dy = 0.$$

From before, we see that A is unique (for each x).

Note that f does not have to be defined at x in order for x to be in the Lebesgue set of f. If $f = g$ a.e., then the Lebesgue set of f coincides with the Lebesgue set of g. If we think of functions in $L^1(\mathbb{R}^n)$ as equivalence classes, then the Lebesgue set of f is well defined.

Lebesgue's theorem. Almost every $x \in \mathbb{R}^n$ is in the Lebesgue set of f, and if $f \colon \mathbb{R}^n \to \mathbb{R}$ is the representation of equivalence classes $[f]$, then $A = f(x)$. For emphasis: If $[f] \in L^1_{\text{loc}}$ is an equivalence class, then for x in the Lebesgue set, $f(x)$ is well defined (defined by the above limit).

For example, let

$$g(x) = \begin{cases} \sin(1/x) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

Then 0 is not in the Lebesgue set of g. Recall: continuous, then in Lebesgue set.

Regular convergence. A sequence of measurable functions $E_1, E_2, \ldots$ *converges regularly* to x if there exists $c > 0$ and $r_1, r_2, \ldots$ such that $E_k \subset B(x, r_k)$, $\lim_{k \to \infty} r_k = 0$, and $\lambda(B(x, r_k)) \leq c\lambda(E_k)$ for all k.

Theorem 0.2. *Let $f \in L^1(\mathbb{R}^n)$, x in the Lebesgue set of f, and $E_1, E_2, \ldots$ converge regularly to x. Then*

$$f(x) = \lim_{k \to \infty} \frac{1}{\lambda(E_k} \int_{E_k} f(y) \ dy.$$

The point here is that we do not have to use balls.

Proof.

$$\begin{aligned} \left| \left(\frac{1}{\lambda(E_k)} \int_{E_k} f(y) \ dy \right) - f(x) \right| &\leq \frac{1}{\lambda(E_k)} \int_{E_k} |f(y) - f(x)| \ dy \\ &\leq \frac{c}{\lambda(B(x, r_k))} \int_{B(x, r_k)} |f(y) - f(x)| \ dy \\ &\to 0 \end{aligned}$$

as $k \to \infty$ since x is in the Lebesgue set of f. $\square$

FTOC II.

Theorem 0.3. *Let $f \in L^1(\mathbb{R})$ and let $a \in \mathbb{R}$. Define*

$$F(x) = \int_a^x f(y)\ dy = \begin{cases} \int_{\mathbb{R}} f(y)\chi_{[a,x]}(y)\ dy & x \geq a; \\ -\int_{\mathbb{R}} f(y)\chi_{[x,a]}(y)\ dy & x < a. \end{cases}$$

Then F is differentiable a.e. and $F' = f$ a.e.

Proof. Almost every $x \in \mathbb{R}$ is in the Lebesgue set of f. We show that $F'(x) = f(x)$ for x in the Lebesgue set of f. By the previous theorem,

$$\lim_{k\to\infty} \frac{1}{\lambda(E_k)} \int_{E_k} f(y)\ dy = f(x)$$

for any regular sequence converging to x. Let $r_k > 0$ such that $\lim r_k = 0$ and $E_k = (x, x + r_k)$. Then E_k regular and

$$\lim_{k\to\infty} \frac{1}{r_k} \int_x^{x+r_k} f(y)\ dy = f(x),$$

i.e.,

$$\lim_{k\to\infty} \frac{F(x + r_k) - F(x)}{r_k} = f(x).$$

Since r_k arbitrary,

$$\lim_{h\to 0^+} \frac{F(x + h) - F(x)}{h} = f(x),$$

and F is right differentiable. Repeat the argument with $E_k = (x - r_k, x)$ and

$$\lim_{h\to 0^-} \frac{F(x + h) - F(x)}{h} = f(x),$$

so F is left and right differentiable and both one-sided derivatives equal $f(x)$. Thus, $F'(x) = f(x)$ for any x in the Lebesgue set, which is almost everywhere by the Lebesgue theorem. $\square$