# MEASURE AND INTEGRATION: LECTURE 13

**Egoroff's theorem (pointwise convergence is nearly uniform.**

**Theorem 0.1.** *Suppose $\mu(X) < \infty$. Let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions such that $f_n \to f$ a.e. For all $\epsilon > 0$, there exists a measurable subset $E \subset X$ with $\mu(X \setminus E) < \epsilon$ and such that $f_n \to f$ uniformly on $E$.*

*Proof.* Let

$$S(n, k) = \bigcap_{i,j>n} \{x \mid |f_i(x) - f_j(x)| < 1/k\}.$$

Clearly, $S(n, k)$ is measurable, since it is the countable intersection of measurable sets. Note that

$$\begin{aligned} S(n, k) &= \bigcap_{i,j>n+1} \{\cdots\} \cap \bigcap_{i=n,j>n+1} \{\cdots\} \\ &= S(n + 1, k) \cap \{\text{stuff}\}. \end{aligned}$$

Thus, $S(n, k) \subset S(n + 1, k)$, that is, for each $k$, we have an ascending sequence of sets. Claim: for each $k$, $X = \cup_{n=1}^{\infty} S(n, k)$. Given $k$, and $x \in X$, we know $f_i(x) \to f(x)$. Thus there exists $N$ such that $|f_i(x) - f_j(x)| < 1/k$ for all $i, j > N$ since any convergent sequence is Cauchy. Thus, $x \in S(N, k)$. Obviously

$$X \supset \bigcup_{n=1}^{\infty} S(N, k) \Rightarrow X = \bigcup_{n=1}^{\infty} S(n, k)$$

for each $k$. So we have

$$\lim_{n\to\infty} \mu(S(n, k)) \to \mu(X)$$

for any $k$.

For each $k = 1, 2, \ldots$, choose $n_k$ so that

$$|\mu(S(n_k, k) - \mu(X)| < \epsilon/2^k.$$

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(Recall $\mu(X) < \infty$.) Let $E = \cap_{k=1}^{\infty} S(n_k, k)$. Then $X \setminus E = \cup_{k=1}^{\infty}(X \setminus S(n_k, k))$. Thus,

$$\begin{aligned}\mu(X \setminus E) &\leq \sum_{k=1}^{\infty} \mu(X \setminus S(n_k, k)) \\ &= \sum_{k=1}^{\infty} \mu(X) - \mu(S(n_k, k)) \\ &\leq \sum_{k=1}^{\infty} |\mu(X) - \mu(S(n_k, k))| \\ &< \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.\end{aligned}$$

Claim: $f_n \to f$ uniformly on $E$; that is, given any $\delta > 0$, there exists $N$ such that $|f_i(x) - f_j(x)| < \delta$ for all $i, j > N$ and *every* $x \in E$. Choose $k$ such that $1/k < \delta$. If $x \in S(n, k)$, by definition $|F_i(x) - f_j(x)| < \delta$ for all $i, j > n$. In particular, for $x \in S(n_k, k)$, $|f_i(x) - f_j(x)| < \delta$ for all $i, j > n_k$. But $S(n_k, k) \supset E$, so $|f_i(x) - f_j(x)| < \delta$ for all $i, j > n_k$ and all $x \in E$. □

The theorem is not necessarily true if $\mu(X) = \infty$. For example, if $\mu$ is Lebesgue measure on $\mathbb{R}$ and $f_n = \chi_{[n,n+1]}$. Then $f_n \to 0$ pointwise, but for any $n \neq m$, $|f_n(x) - f_m(x)| = 1$ on a set of measure 2.

**Convergence in measure.** Here is an example. Let $f_n \colon X \to \mathbb{R}$ and $\int_X |f_n| \to 0$. Then if $\epsilon > 0$,

$$int_X |f_n| \geq \int_{\{x \ | \ f_n > \epsilon\}} |f_n| > \epsilon\mu(\{x \mid f_n(x) > \epsilon\}).$$

So, given $\epsilon > 0$, choose $N$ such that for all $n > N$, $\int_X |f_n| < \epsilon^2$. Then $\epsilon \geq \mu(\{x \mid f_n(x) > \epsilon\}$ for all $n > N$.

We say that $f_n \to f$ *in measure* if given $\epsilon > 0$ there exists $N$ such that, for all $n \geq N$, $\mu(\{x \mid |f(x) - f_n(x)| > \epsilon\}) < \epsilon$.

**Convergence almost everywhere implies convergence in measure.**

**Theorem 0.2.** *If $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ is measure.*

*Proof.* Let $A = \{x \mid f_n(x) \to f(x)\}$. Then $\mu(X \setminus A) = 0$. Since $\mu(X) < \infty$, $\mu(A) < \infty$ and we may apply Egoroff's theorem. Thus, there exists a set $E$ such that $\mu(A \setminus E) < \epsilon$ and $f_n \to f$ uniformly on $E$. Given $\epsilon > 0$, there exists $N$ such that $|f(x) - f_n(x)| < \epsilon$ for all

$n > N$ and all $x \in E$. So, for $n > N$, $|f(x) - f_n(x)|$ can be greater than $\epsilon$ only on $(A \setminus E) \cup (X \setminus A)$. This means that

$$\begin{aligned} \mu(\{x \mid |f_n(x) - f(x)| > \epsilon\}) &\leq \mu(A \setminus E) + \mu(X \setminus A) \\ &< \epsilon + 0 = \epsilon \end{aligned}$$

for all $n > N$. □

However, if $f_n \to f$ in measure, then it is *not true* that $f_n \to f$ a.e. For example, $f_n \colon [0,1] \to [0,1]$ such that $\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = 0$ but $f_n(x) \to 0$ for no $x$.

**Convergence in measure implies some subsequence convergence almost everywhere.**

**Theorem 0.3.** *If $f_n \to f$ in measure, then $f_n$ has a subsequence $f_{n_k}$ such that $\lim_{k\to\infty} f_{n_k} = f$ a.e.*

*Proof.* Let $\epsilon = 2^{-k}$. Given $k$, there exists $N(k)$ such that for $n \geq N(k)$, $\mu(\{x \mid |f(x) - f_n(x)| > 2^{-k}\} < 2^{-k}$. Let $E_k = \{x \mid \left|f_{N(k)}(x) - f(x)\right| > 2^{-k}\}$. Then $\mu(E_k) < 2^{-k}$. If $x \notin \cup_{i=k}^\infty E_i$, then $x \in (\cup_{i=k}^\infty E_i)^c = \cap_{i=k}^\infty E_i^c$. Then

$$\begin{aligned} \left|f_{N(i)}(x) - f(x)\right| &< 2^{-i} \quad \text{for every } i \geq k \\ &\Rightarrow f_{N(i)}(x) \to f(x). \end{aligned}$$

Let

$$A = \bigcap_{k=1}^\infty \bigcup_{i=k}^\infty E_i.$$

So if $x \notin A$, then $f_{N(i)}(x) \to f(x)$. For any $k$,

$$\mu(A) \leq \mu\left(\cup_{i=k}^\infty E_i\right) \leq \sum_{i=k}^\infty 2^{-i} = 2^{-k+1},$$

so $\mu(A) = 0$. □

**Dominated convergence theorem holds for convergence in measure.** We know that dominated convergence and monotone convergence still hold if we replace convergence with convergence almost everywhere. Now we show that the theorems are valid if we replace convergence by convergence in measure.

**Theorem 0.4.** *Let $f_n \colon X \to \mathbb{C}$ be a sequence of measurable functions defined a.e. Suppose $f_n \to f$ in measure and $|f_n| \leq |g|$ a.e. with $g \in L^1(\mu)$. Then*

$$\int_X f\ d\mu = \lim_{n\to\infty} f_n\ d\mu.$$

Note that the pointwise limit of $f_n$ may not exist.

*Proof.* Take any subsequence $f_{n_k}$. Clearly, $f_{n_k} \to f$ in measure. There exists a subsequence $f_{(n_k)_\ell}$ such that $f_{(n_k)_\ell} \to f$ pointwise a.e. Apply dominated convergence to this subsequence. Then

$$\int_X f \, d\mu = \lim_{\ell \to \infty} \int_X f_{(n_k)_\ell} \, d\mu.$$

$\square$

**Lemma 0.5.** *Let $a_n$ be a sequence. If every subsequence has a subsequence which converges to $\alpha$, then $\lim_{n\to\infty} a_n = \alpha$.*