Lecture 24

Proposition. $L^t = *^{-1}L*$

Proposition. *$u \in V$ then $[L_u^t, L] = -L_u$.*

Proof. Proof omitted. □

Let (X^{2n}, ω) be a compact symplectic manifold. Let $x \in X$ and $V = T_x^*$. Notice

(a) From ω_x we get a symplectic bilinear form on T_x.

(b) From this form we get an identification $T_x \to T_x^*$.

(c) Hence from $1, 2$ we get a symplectic bilinear from B_x on V.

(d) From B_x we get a $*$-operator
$$*_x : \Lambda^p(T_x^*) \to \Lambda^{2n-p}(T_x^*)$$

(e) This gives us a $*$-operator on forms
$$* : \Omega^p(X) \to \Omega^{2n-p}(X)$$

We can define a symplectic version of the L^2 inner product on Ω^p as follows. Take $\alpha, \beta \in \Omega^p$ and define
$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta$$

(Note: This is not positive definite or anything, its just a pairing)

Take $\alpha \in \Omega^{p-1}$, $\beta \in \Omega^p$. Then look at
$$\begin{aligned} d(\alpha \wedge *\beta) &= d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d * \beta \\ &= d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge *(*^{-1}d*)\beta \end{aligned}$$

Since $\int_X d(\alpha \wedge *\beta) = 0$, we integrate both sides of the above and get
$$\int_X d\alpha \wedge *\beta = (-1)^p \int \alpha \wedge *(*^{-1}d*)\beta$$
If we introduce the notation $\delta = (-1)^p *^{-1} d*$ on Ω^p then
$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$$
Now, given the mapping $L : \Omega^p \to \Omega^{p+2}$, $L\alpha = \omega \wedge \alpha$ we have the following theorem

Theorem. $[\delta, L] = d$.

This identity has no analogue in ordinary Hodge Theory. This is very important.

Proof. $x \in X, \xi \in T_x^*$, then $\sigma(d)(x, \xi) = iL_\xi$. On Λ^p, $\delta = (-1)^p *^{-1} d*$, so $\sigma(d)(x,\xi) = (-1)^p i *^{-1} L_\xi * = -iL_\xi^t$. Then
$$\sigma([\delta, L]) = i[L_\xi^t, L] = iL_\xi = \sigma(d)(x, \xi)$$
so $[\delta, L]$ and d have the same symbol.

Now, d $[\delta, L]$ are first order DO's mapping $\Omega^p \to \Omega^{p+1}$, so $d - [\delta, L] : \Omega^p \to \Omega^{p+1}$ is a first order DO. We want to show that this is 0.

Let $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a Darboux coordinate patch. Consider $u = \beta_1 \wedge \cdots \wedge \beta_n$ where $\beta_i = 1, dx_i, dy_i$ or $dx_i \wedge dy_i$.

These de Rham forms are a basis at each point of $\Lambda(T_x^*)$.

$Lu = \omega \wedge u$ is again a form of this type since $\omega = \sum dx_i \wedge dy_i$ is of this form. Also $*u$ i s of this from.

Note that $d = 0$ on a form of this type, hence $\delta = *^{-1}d*$ is 0 on a form of this type. Thus $[\delta, L] - d$ is 9 on a form of this type. □