Lecture 21

The Hodge $*$-operator

Let $V = V^n$ be an n-dimensional $\mathbb{R}$-vector space. Let $B : V \times V \to \mathbb{R}$ be a non-degenerate bilinear form on V (Note that for the momentum we are not assuming anything about this form).

From B one gets a non-degenerate bilinear form $B : \Lambda^k(V) \times \lambda^k(V) \to \mathbb{R}$. If $\alpha = v_1 \wedge \cdots \wedge v_k, \beta = w_1 \wedge \cdots \wedge w_k$ then

$$B(\alpha, \beta) = \det(B(v_i, v_j))$$

Alternate definition:

Define a pairing (non-degenerate and bilinear) $\Lambda^k(V) \times \Lambda^k(V^*) \to \mathbb{R}$ with $\alpha = v_1 \wedge \cdots \wedge v_k$, $\beta = f_1 \wedge \cdots \wedge f_k$, $v_i \in V$, $f_i \in V^*$. Then

$$\langle \alpha, \beta \rangle = d \langle v_i, f_j \rangle$$

This gives rise to the identification $\Lambda^k(V^*) \cong \Lambda^k(V)^*$.

So $B : V \times V \to \mathbb{R}$ gives to $L_B : V \xrightarrow{\cong} V^*$ by $B(u, v) = \langle u, L_B v \rangle$. This can be extended to a map of k-th exterior powers, $L_B : \Lambda^k(V) \to \Lambda^k(V^*)$, defined by

$$L_B(v_1 \wedge \cdots \wedge v_k) = L_B v_1 \wedge \cdots \wedge L_B v_k$$

and if we have $\alpha, \beta \in \Lambda^k(V)$ then $B(\alpha, \beta) = \langle \alpha, L_B \beta \rangle$.

Let us now look at the top dimensional piece of the exterior algebra. $\dim \Lambda^n(V) = 1$, orient V so that we are dealing with $\Lambda^k(V)_+$. Then there is a unique $\Omega \in \Lambda^n(V)$ such that $B(\Omega, \Omega) = 1$.

Theorem. *There exists a bijective map* $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$ *such that for* $\alpha, \beta \in \Lambda^k(V)$ *we have*

$$\alpha \wedge * \beta = B(\alpha, \beta)\Omega$$

Proof. From Ω we get a map $\Lambda^n(V) \xrightarrow{\cong} \mathbb{R}$, $\lambda\Omega \mapsto \lambda$. So we get a non-degenerate pairing

$$\Lambda^k(V) \times \Lambda^k(V) \to \Lambda^n(V) \to \mathbb{R}$$

Now we have a mapping $\Lambda^k(V^*) \xrightarrow{k} \Lambda^{n-k}(V)$. Define the $*$-operator to be $k \circ L_B$.

Multiplicative Properties of $*$

There are actually almost no multiplicative properties of the $*$-operator, but there are a few things to be said.

Suppose we have a vector space $V^n = V_1^{n_1} \oplus V_2^{n_2}$ and suppose we have the bilinear form $B = B_1 \oplus B_2$. From this decomposition we can split the exterior powers

$$\Lambda^k(V) = \bigoplus_{r+s=k} \Lambda^r(V_1) \otimes \Lambda^s(V_2)$$

If $\alpha_1, \beta_1 \in \Lambda^r(V_1)$ and $\alpha_2, \beta_2 \in \Lambda^r(V_2)$ then

$$B(\alpha_1 \wedge \alpha_2, \beta_1 \wedge \beta_2) = B_1(\alpha_1, \beta_1) B_2(\alpha_2, \beta_2)$$

Theorem. *With* $\beta_1 \in \Lambda^r(V_1)$ *and* $\beta_2 \in \Lambda^s(V_2)$ *we have*

$$*(\beta_1 \wedge \beta_2) = (-1)^{(n_1 - r)s} *_1 \beta_1 \wedge *_2 \beta_2$$

Proof. $\alpha_1 \in \Lambda^r(V_1)$, $\alpha_2 \in \Lambda^s(V_2)$ with Ω_1, Ω_2 the volume forms on the vector spaces. Then let $\Omega = \Omega_1 \wedge \Omega_2$ be the volume form for $\Lambda^n(V)$. Then

$$\begin{aligned}(\alpha_1 \wedge \alpha_2) * (\beta_1 \wedge \beta_2) &= B(\alpha_1 \wedge \alpha_2, \beta_1 \wedge \beta_2)\Omega = B_1(\alpha_1, \beta_1)\Omega_1 \wedge B(\alpha_2, \beta_2)\Omega_2 \\ &= (\alpha_1 \wedge *_1\beta_1) \wedge (\alpha_2 \wedge *_2\beta_2) \\ &= (-1)^{(n_1 - r)s} \alpha_1 \wedge \alpha_2 \wedge (*_1\beta_1 \wedge *_2\beta_2)\end{aligned}$$

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