Lecture 4

Applying Hartog's Theorem

Let $X \subset \mathbb{C}^n$ be an algebraic variety, $\operatorname{cod}_{\mathbb{C}} X = 2$. And suppose $f \in \mathcal{O}(\mathbb{C}^n - X)$. Then f extends holomorphically to $f \in \mathcal{O}(\mathbb{C}^n)$.

Sketch of Proof: Cut X by a complex plane ($P = \mathbb{C}^2$) transversally. Then $f \mid_P \in \mathcal{O}(P - \{p\})$ so by hartog, $f \mid_P \in \mathcal{O}(P)$. Do this argument for all points, so f has to be holomorphic on $f \in \mathcal{O}(\mathbb{C}^n)$.

We have to be a little more careful to actually prove it, but this is just an example of how algebraic geometers use this.

Dolbeault Complex and the ICR Equation

Let U be an open subset of $\mathbb{C}^n$, $\omega \in \Omega^1(U)$, then we discussed how $\Omega^1(U) = \Omega^{1,0} \oplus \Omega^{0,1}$.

There is a similar story for higher degree forms.

Take $r > 1$, $p + q = r$. Then $\omega \in \Omega^{p,q}(U)$ if ω is in the following form

$$\omega = \sum f_{I,J} dz_I \wedge d\bar{z}_J \qquad f_{I,J} \in C^\infty(U)$$

and $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ are standard multi-indices. Then

$$\Omega^r = \bigoplus_{p+q=r} \Omega^{p,q}(U)$$

Now suppose we have $\omega \in \Omega^{p,q}(U)$, $\omega = \sum f_{I,J} dz_I \wedge d\bar{z}_J$ then the de Rham differential is written as follows

$$dw = \sum df_{IJ} \wedge dz_I \wedge dz_J = \sum \frac{\partial f_{I,J}}{\partial z_i} dz_i \wedge dz_I \wedge dz_J + \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J$$

The first term we define to be $\partial\omega$ and the second to be $\bar{\partial}\omega$,i.e.

$$\partial\omega = \sum \frac{\partial f_{I,J}}{\partial z_i} dz_i \wedge dz_I \wedge dz_J$$
$$\overline{\partial}\omega = \sum \frac{\partial f_{I,J}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J$$

Now we may write $d\omega = \partial\omega + \overline{\partial}\omega$, and note that $\partial\omega \in \Omega^{p+1,q}(U)$ and $\overline{\partial}\omega \in \Omega^{p,q+1}(U)$.

Also

$$d^2 = 0 = \partial^2\omega + \partial\overline{\partial}\omega + \overline{\partial}\overline{\partial}\omega + \overline{\partial}^2\omega$$

and the terms in the above expression are of bidegree

$$(p+2, q) + (p+1, q+1) + (p+1, q+1_+(p, q+2)$$

so $\overline{\partial}^2 = \partial^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$, so $\partial, \overline{\partial}$ are anti-commutative.

We now have that the de Rham complex $(\Omega^*(U), d)$ is a bicomplex, i.e. d splits into two different coboundary operators that anticommute.

The rows of the bicomplex are given by

$$\Omega^{0,q} \xrightarrow{\partial} \Omega^{1,q} \xrightarrow{\partial} \Omega^{2,q} \xrightarrow{\partial} \cdots$$

and the columns are given by

$$\Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \xrightarrow{\overline{\partial}} \cdots$$

For the moment, we focus on the columns, more specifically the extreme left column.

Definition. The **Dolbeault Complex** is the following complex

$$C^\infty(U) = \Omega^0 = \ \Omega^{0,0}(U) \xrightarrow{\overline{\partial}} \Omega^{0,1}(U) \xrightarrow{\overline{\partial}} \Omega^{0,2}(U) \xrightarrow{\overline{\partial}} \cdots$$

A basic problem in several complex variables is to answer the question: For what open sets U in $\mathbb{C}^n$ is this complex exact?

Today we will show that the Dolbeault complex is locally exact (actually, we will prove something a little stronger)

Theorem (1). *Let U and V be polydisks with $\overline{V} \subset U$. Then if $\omega \in \Omega^{0,q}(U)$ and $\overline{\partial}\omega = 0$ then there exists $\mu \in \Omega^{0,q-1}(V)$ with $\overline{\partial}\mu = \omega$ on V.*

This just says that if we shrink the domain a little, the exactness holds.

To prove this theorem we will use a trick similar to showing that the real de Rham complex is locally exact.

First, we define a new set

Definition. $\Omega^{0,q}(U)_k$, $0 \le k \le n$ is given by the following rule: $\omega \in \Omega^{0,q}(U)_k$ if and only if

$$\omega = \sum f_I d\bar{z}_I \qquad d\bar{z}_I = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}, \quad 1 \le i_1 \le \cdots \le i_q \le k$$

This is just a restriction on the $\bar{z}_j$'s that may be present. For example $\Omega^{0,q}(U)_0 = \{0\}$ and $\Omega^{0,q}(U)_n = \Omega^{0,q}(U)$.

An important property of this space follows. If $\omega \in \Omega^{0,q}(U)_k$ then

$$\overline{\partial}\omega = \sum_{l>k} \frac{\partial f_I}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_I + \Omega^{0,q+1}(U)_k$$

so if $\overline{\partial}\omega = 0$ then $\partial f_I / \partial \bar{z}_l = 0$, for $l > k$ i.e. f_I is holomorphic.

Let V, U be polydisks, $\overline{V} \subset U$. Choose a polydisk W so that $\overline{V} \subset W$ and $\overline{W} \subset U$.

Theorem (2). *If $\omega \in \Omega^{0,q}(U)_k$ and $\overline{\partial}\omega = 0$ then there exists $\beta \in \Omega^{0,q-1}(W)_{k-1}$ such that $\omega - \overline{\partial}\beta \in \Omega^{0,q}(W)_{k-1}$.*

We claim that Theorem 2 implies Theorem 1 (left as exercise)

Before we prove theorem 2, we need a lemma

Lemma. *(ICR in 1D) If $g \in C^\infty(U)$ with $\frac{\partial g}{\partial \bar{z}_l} = 0$, $l > k$ then there exists $f \in C^\infty(W)$ such that $\frac{\partial f}{\partial \bar{z}_l} = 0$ for $l > k$ and $\frac{\partial f}{\partial \bar{z}_k} = g$.*

Proof. $U = U_1 \times \cdots \times U_n$ where U_i are disks and $W = W_1 \times \cdots \times W_n$ where W_i are disks. Let $\rho \in C_0^\infty(U_k)$ so that $\rho \equiv 1$ on a neighborhood of $\overline{W}_k$. Replacing g by $\rho(z_k)g$ we can assume that g is compactly supported in z_k.

Choose f to be

$$f = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(z_1, \ldots, z_{k-1}, \eta, z_{k+1}, \ldots, z_n) d\eta \wedge d\bar{\eta}}{\eta - z_k}$$

We showed before that $\frac{\partial f}{\partial \bar{z}_k} = g$. By a change of variable we see that

$$f = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(z_1, \ldots, z_{k-1} z_k - \eta, z_{k+1}, \ldots, z_n)}{\eta} d\eta \wedge d\bar{\eta}$$

so $f \in C^\infty(W)$ and clearly $\frac{\partial f}{\partial \bar{z}_l} = 0$, $l > k$. QED. $\square$

We may now prove Theorem 2

Proof of Theorem 2. $\omega \in \Omega^{0,q}(U)_k$, and $\overline{\partial}\omega = 0$. Write

$$\omega = \mu + d\bar{z}_k \wedge \nu \qquad \mu \in \Omega^{0,q}(U)_{k-1}, \nu \in \Omega^{0,q-1}(U)_{k-1}$$

(just decompose ω) and say

$$\nu = \sum g_I d\bar{z}_I, \qquad g_I \in C^\infty(U), \quad I = (i_1, \dots, i_{q-1}), \quad i_s \leq k-1$$

$\overline{\partial}\omega = 0$ tells use that $\frac{\partial g_I}{\partial \bar{z}_l} = 0$, $l > k$. By the lemma above, there exists $f_I \in C_0^\infty(W)$ so that

$$\frac{\partial f_I}{\bar{z}_k} = g_I \qquad \text{and} \qquad \frac{\partial f_I}{\partial \bar{z}_l} = 0, \; l > k$$

Take $\beta = \sum f_I dz_I$, then

$$\overline{\partial}\beta = \sum d\bar{z}_k \wedge \frac{\partial f_I}{\partial \bar{z}_k} dz_i + \Omega^{0,q}(W)_{k-1} = dz_k \wedge \nu$$

so $\omega - \overline{\partial}\beta \in \Omega^{0,q}(W)_{k-1}$.

$\square$

Theorem (3). *Let U be a polydisk then the Dolbeault complex*

$$\Omega^{0,0}(U) \xrightarrow{\overline{\partial}} \Omega^{0,1}(U) \xrightarrow{\overline{\partial}} \Omega^{0,2}(U) \xrightarrow{\overline{\partial}} \cdots$$

is exact. That is, you don't have to pass to sub-polydisks.

The above theorem is **EXERCISE 1**