18.112 Functions of a Complex Variable
Fall 2008

Lecture 19: Normal Families

(Replacing Text 219-227)

Theorem 1 *Let $\Omega \subset \mathbb{C}$ be a region, $\mathcal{F}$ a family of holomorphic functions on Ω such that for each compact $E \subset \Omega$, $\mathcal{F}$ is uniformly bounded on E. Then $\mathcal{F}$ has a subsequence converging uniformly on each compact subset of Ω.*

First we prove that on each compact subset $E \subset \Omega$, the family $\mathcal{F}$ is equicontinuous. This means, given $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$,

$$|f(z') - f(z'')| < \epsilon \qquad \text{if } |z' - z''| < \delta,\ z', z'' \in E. \tag{1}$$

The distance function $x \to d(x, \mathbb{C} - \Omega)$ is continuous and has a minimum > 0 on the compact set E. Let $d > 0$ be such that (D denoting disk) $F = \bigcup_{x \in E} D(x, 2d)$ has closure $\bar{F} \subset \Omega$.

Let $z', z'' \in E$ satisfy

$$|z' - z''| < d$$

and let γ denote the circle

$$\gamma \ :\ |z - z'| = 2d.$$

Then $\gamma \subset \bar{F}$ and z' and z'' are both inside γ. Also $|\zeta - z'| = 2d$, $|\zeta - z''| \geq d$ for $\zeta \in \gamma$.
By Cauchy's formula for $f \in \mathcal{F}$,

$$f(z') - f(z'') = \frac{z' - z''}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z')(\zeta - z'')} d\zeta,$$

so if $M(\bar{F})$ is the maximum of f on $\bar{F}$

$$|f(z') - f(z'')| \leq |z' - z''| \frac{M(\bar{F})}{d}.$$

Hence (1) follows.

To conclude the proof of Theorem 1 choose any sequence (z_j) which is dense in Ω. Let f_m be any sequence in $\mathcal{F}$. The sequence $f_m(z_1)$ is bounded so f_m has a subsequence $f_{m,1}$ converging at z_1. Form this take a subsequence $f_{m,2}$ which converges at z_2. Continuing we see that the subsequence $f_{m,m}$ converges at each z_j.

By the first part of the proof, $\mathcal{F}$ is equicontinuous on the compact set $\bar{F}$. Given $\epsilon > 0$ there exists a $\delta < d$ such that (1) holds for all $z', z'' \in \bar{F},\ f \in \mathcal{F}$. If $z \in E$ the disk $D(z, \delta)$ contains some z_j so $D(z_j, \delta)$ contains z.

By the compactness of E,

$$E \subset \bigcup_{i=1}^{p} D(z_i, \delta)$$

for some $z_1, \cdots, z_p$. Thus given $z \in E$ there exists a $z_i = z_i(z)$ such that $|z - z_i(z)| < \delta$. Then $z_i(z) \in \bar{F}$. Thus by (1) for $\bar{F}$,

$$|f(z) - f(z_i(z))| < \epsilon. \qquad f \in \mathcal{F}. \tag{2}$$

There exists $N > 0$ such that

$$|f_{r,r}(z_i) - f_{s,s}(z_i)| < \epsilon \qquad 1 \leq i \leq p,\ r, s > N. \tag{3}$$

Given $z \in E$ we have with $z_i = z_i(z)$

$$\begin{aligned}|f_{r,r}(z) - f_{s,s}(z)| &\leq |f_{r,r}(z) - f_{r,r}(z_i)| + |f_{r,r}(z_i) - f_{s,s}(z_i)| + |f_{s,s}(z_i) - f_{s,s}(z)| \\ &\leq 3\epsilon \text{ by (2) and (3).}\end{aligned}$$

The proves the stated uniform convergence on E.

<u>Remark:</u> In the text, p. 223, it is erroneously assumed (and used) that $\zeta_k \in E$. This error occurs in many other texts.