18.112 Functions of a Complex Variable
Fall 2008

# Lecture 15: Contour Integration and Applications

(Text 154-161)

## Remarks on Lecture 15

In parts 4 and 5 (p. 154-160) some clarification of the use of the logarithm are called for.

### Example 4 p.159

The relation

$$(-z)^{2\alpha} = e^{2\pi i\alpha} z^{2\alpha}$$

which is crucial for proof deserves explanation.

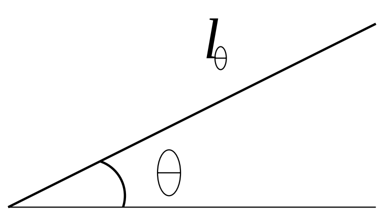


Fig. 15-1

We consider the function

$$\log_\theta z = \log |z| + i\arg_\theta z$$

in the region $\mathbb{C} - l_\theta$ (the plane with the ray $l_\theta$ removed) where the angle is fixed by

$$\theta < \arg_\theta z < \theta + 2\pi.$$

In the problem of computing

$$\int_0^\infty x^\alpha R(x)\, dx$$

we consider

$$\log_{-\frac{\pi}{2}}(z)$$

in the plane $\mathbb{C}$ with the negative imaginary axis removed and us the Residue theorem on the contour in Fig. 4.13. As in the text we arrive at the integral

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2)\, dz = \int_0^{\infty} \left(z^{2\alpha+1} + (-z)^{2\alpha+1}\right) R(z^2)\, dz.$$

On the right $z$ belongs to $(0, \infty)$ and

$$\begin{aligned}
\log_{-\frac{\pi}{2}}(z) &= \log|z| + \left(-\frac{\pi}{2} + \frac{\pi}{2}\right) i, \qquad \frac{-\pi}{2} < \arg_{-\frac{\pi}{2}} z < \frac{3\pi}{2},\\
\log_{-\frac{\pi}{2}}(-z) &= \log|z| + \left(-\frac{\pi}{2} + \frac{3\pi}{2}\right) i\\
&= \log_{-\frac{\pi}{2}}(z) + i\pi, \qquad z > 0.
\end{aligned}$$

Thus for $z > 0$,

$$\begin{aligned}
(-z)^{2\alpha+1} &= e^{(2\alpha+1)\log_{-\frac{\pi}{2}}(-z)}\\
&= e^{(2\alpha+1)(\log|z| + i\pi)}\\
&= -e^{2\alpha i\pi} z^{2\alpha+1},
\end{aligned}$$

so the last integrals combine to

$$(1 - e^{2\alpha i\pi}) \int_0^{\infty} z^{2\alpha+1} R(z^2)\, dz.$$

For $z > 0$ we have from the above

$$\log_{-\frac{\pi}{2}}(z) = \log|z|,$$

so

$$\begin{aligned}
\frac{1}{2\pi i}\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2)\, dz &= \frac{1}{2\pi i}(1 - e^{2\alpha i\pi}) \int_0^{\infty} x^{2\alpha+1} R(x^2)\, dx\\
&= -\frac{1}{\pi} e^{\alpha\pi i} \sin\pi\alpha \int_0^{\infty} x^{2\alpha+1} R(x^2)\, dx.
\end{aligned} \tag{1}$$

The left hand side of (1) is the sum of the residues of

$$z^{2\alpha+1} R(z^2) = f(z)$$

in the upper half plane. If

$$R(z^2) = \frac{g(z)}{h(z)},$$

where $g$ and $h$ are holomorphic, $g(a) \neq 0$, and $h$ has a simple zero at $a$, then

$$\mathrm{Res}_{z=a} f(z) = z^{(2\alpha+1)}(a) \frac{g(a)}{h'(a)}, \tag{2}$$

where

$$z^{2\alpha+1} = e^{(2\alpha+1)\log_{-\frac{\pi}{2}}(z)}.$$

**Example: Exercise 3(g) p.161**

To calculate

$$\int_0^\infty x^{\frac{1}{3}} \frac{dx}{1+x^2},$$

we use $x = t^2$ and arrive at

$$\int_{-\infty}^\infty z^{\frac{5}{3}} \frac{dz}{1+z^4}$$

in (1). The poles in the upper half plane are

$$z = e^{i\frac{\pi}{4}} \qquad \text{and} \qquad z = e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)}.$$

We use (2) to calculate the residues:

$$\begin{aligned}
\text{Res}_{z=e^{i\frac{\pi}{4}}} \left( z^{\frac{5}{3}} \frac{1}{1+z^4} \right) &= z^{\frac{5}{3}} \left( e^{i\frac{\pi}{4}} \right) \frac{1}{4(e^{i\frac{\pi}{4}})^3} \\
&= e^{\frac{5}{3} \log_{-\frac{\pi}{2}} \left( e^{i\frac{\pi}{4}} \right)} \frac{1}{4(e^{i\frac{\pi}{4}})^3} \\
&= e^{\frac{5}{3}\left(i\frac{\pi}{4}\right)} \frac{1}{4(e^{i\frac{\pi}{4}})^3} \\
&= \frac{1}{4} e^{-i\frac{\pi}{3}},
\end{aligned}$$

and

$$\begin{aligned}
\text{Res}_{z=e^{i\frac{3\pi}{4}}} \left( z^{\frac{5}{3}} \frac{1}{1+z^4} \right) &= z^{\frac{5}{3}} \left( e^{i\frac{3\pi}{4}} \right) \frac{1}{4(e^{i\frac{3\pi}{4}})^3} \\
&= e^{\frac{5}{3} \log_{-\frac{\pi}{2}} \left( i\left(-\frac{\pi}{2}+\frac{5\pi}{4}\right)\right)} \frac{1}{4(e^{i\frac{3\pi}{4}})^3} \\
&= \frac{1}{4} e^{-i\pi}.
\end{aligned}$$

Thus (1) gives

$$\frac{1}{4} e^{-i\frac{\pi}{3}} + \frac{1}{4} e^{-i\pi} = -\frac{1}{\pi} e^{\frac{1}{3}\pi i} \sin\frac{\pi}{3} \int_0^\infty x^{\frac{5}{3}} \frac{dx}{1+x^4},$$

so

$$\int_0^\infty x^{\frac{5}{3}} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{3}}.$$

**Example 5 p.160**

The last four lines on the page are a bit misleading because the specific logarithm has already been chosen. So here is a completion of the proof after the equation

$$\int_0^\pi \mathrm{Log}(-2ie^{ix}\sin x)\ dx = 0.$$

We know (Lecture 2) that

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}z_1 + \mathrm{Log}z_2, \qquad \text{if } -\pi < \mathrm{Arg}z_1 + \mathrm{Arg}z_2 < \pi. \tag{3}$$

Using this for $z = 2\sin x$ we get

$$\int_0^\pi \log(2\sin x)\ dx + \int_0^\pi \mathrm{Log}(-ie^{ix})\ dx = 0. \tag{4}$$

But

$$\mathrm{Log}(-i) = -\frac{\pi i}{2}, \qquad \mathrm{Log}e^{ix} = ix\ (0 < x < \pi),$$

so since $-\frac{\pi}{2} + x$ is in $(-\pi, \pi)$, (3) implies

$$\mathrm{Log}(-ie^{ix}) = -\frac{\pi i}{2} + ix.$$

Now (4) implies the result

$$\int_0^\pi \log\sin\theta\ d\theta = -\pi\log 2.$$