18.102 Introduction to Functional Analysis
Spring 2009

Problem 8.1 Show that a continuous function $K : [0,1] \longrightarrow L^2(0,2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0,2\pi)$, for $x \in [0,1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \le n$ then $K_n : [0,1] \longrightarrow L^2(0,2\pi)$ is also continuous and

$$\sup_{x\in[0,1]} \|K(x) - K_n(x)\|_{L^2(0,2\pi)} \to 0. \tag{18.8}$$

Hint. Use one of the properties of compactness in a Hilbert space that you proved earlier.

Problem 8.2

Consider an integral operator acting on $L^2(0,1)$ with a kernel which is continuous – $K \in \mathcal{C}([0,1]^2)$. Thus, the operator is

$$Tu(x) = \int_{(0,1)} K(x,y)u(y). \tag{18.9}$$

Show that T is bounded on L^2 (I think we did this before) and that it is in the norm closure of the finite rank operators.

Hint. Use the previous problem! Show that a continuous function such as K in this Problem defines a continuous map $[0,1] \ni x \longmapsto K(x,\cdot) \in \mathcal{C}([0,1])$ and hence a continuous function $K : [0,1] \longrightarrow L^2(0,1)$ then apply the previous problem with the interval rescaled.

Here is an even more expanded version of the hint: You can think of $K(x,y)$ as a continuous function of x with values in $L^2(0,1)$. Let $K_n(x,y)$ be the continuous function of x and y given by the previous problem, by truncating the Fourier series (in y) at some point n. Check that this defines a finite rank operator on $L^2(0,1)$ – yes it maps into continuous functions but that is fine, they are Lebesgue square integrable. Now, the idea is the difference $K - K_n$ defines a bounded operator with small norm as n becomes large. It might actually be clearer to do this the other way round, exchanging the roles of x and y.

Problem 8.3 Although we have concentrated on the Lebesgue integral in one variable, you proved at some point the covering lemma in dimension 2 and that is pretty much all that was needed to extend the discussion to 2 dimensions. Let's just assume you have assiduously checked everything and so you know that $L^2((0,2\pi)^2)$ is a Hilbert space. Sketch a proof – noting anything that you are not sure of – that the functions $\exp(ikx + ily)/2\pi$, $k, l \in \mathbb{Z}$, form a complete orthonormal basis.