Lecture 8

3.2 Riemann Integral of Several Variables

Last time we defined the Riemann integral for one variable, and today we generalize to many variables.

Definition 3.3. A *rectangle* is a subset Q of $\mathbb{R}^n$ of the form

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n], \tag{3.10}$$

where $a_i, b_i \in \mathbb{R}$.

Note that $x = (x_1, \ldots, x_n) \in Q \iff a_i \leq x_i \leq b_i$ for all i. The volume of the rectangle is

$$v(Q) = (b_1 - a_1) \cdots (b_n - a_n), \tag{3.11}$$

and the width of the rectangle is

$$\text{width}(Q) = \sup_i (b_i - a_i). \tag{3.12}$$

Recall (stated informally) that given $[a, b] \in \mathbb{R}$, a finite subset P of $[a, b]$ is a partition of $[a, b]$ if $a, b \in P$ and you can write $P = \{t_i : i = 1, \ldots, N\}$, where $t_1 = a < t_2 < \ldots < t_N = b$. An interval I belongs to P if and only if I is one of the intervals $[t_i, t_{i+1}]$.

Definition 3.4. A *partition P of Q* is an n-tuple $(P_1, \ldots, P_n)$, where each P_i is a partition of $[a_i, b_i]$.

Definition 3.5. A rectangle $R = I_1 \times \cdots \times I_n$ *belongs to* P if for each i, the interval I_i belongs to P_i.

Let $f : Q \to \mathbb{R}$ be a bounded function, let P be a partition of Q, and let R be a rectangle belonging to P.

We define

$$\begin{aligned} m_R f &= \inf_R f = \text{g.l.b. } \{f(x) : x \in \mathbb{R}\} \\ M_R f &= \sup_R f = \text{l.u.b. } \{f(x) : x \in \mathbb{R}\}, \end{aligned} \tag{3.13}$$

from which we define the lower and upper Riemann sums,

$$\begin{aligned} L(f, P) &= \sum_R m_R(f) v(R) \\ U(f, P) &= \sum_R M_R(f) v(R). \end{aligned} \tag{3.14}$$

It is evident that

$$L(f, P) \leq U(f, P). \tag{3.15}$$

Now, we will take a sequence of partitions that get finer and finer, and we will define the integral to be the limit.

Let $P = (P_1, \ldots, P_n)$ and $P' = (P'_1, \ldots, P'_n)$ be partitions of Q. We say that P' refines P if $P'_i \supset P_i$ for each i.

Claim. *If P' refines P, then*

$$L(f, P') \geq L(f.P). \tag{3.16}$$

Proof. We let $P_j = P'_j$ for $j \neq i$, and we let $P'_i = P_i \cup \{a\}$, where $a \in [a_i, b_i]$. We can create any refinement by multiple applications of this basic refinement. If R is a rectangle belonging to P, then either

1. R belongs to P', or
2. $R = R' \cup R''$, where R', R'' belong to P'.

In the first case, the contribution of R to $L(f, P')$ equals the contribution of R to $L(f, P)$, so the claim holds.

In the second case,

$$m_R v(R) = m_R(v(R') + v(R'')) \tag{3.17}$$

and

$$m_r = \inf_R f \leq \inf_{R'} f, \inf_{R''} f. \tag{3.18}$$

So,

$$m_R \leq m_{R'}, m_{R''} \tag{3.19}$$

Taken altogether, this shows that

$$m_R v(R) \leq m_{R'} v(R') + m_{R''} v(R'') \tag{3.20}$$

Thus, R' and R'' belong to P'. □

Claim. *If P' refines P, then*

$$U(f, P') \leq U(f, P) \tag{3.21}$$

The proof is very similar to the previous proof. Combining the above two claims, we obtain the corollary

Corollary 2. *If P and P' are partitions, then*

$$U(f, P') \geq L(f, P) \tag{3.22}$$

Proof. Define $P'' = (P_1 \cup P'_1, \dots, P_n \cup P'_n)$. So, P'' refines P and P'. We have shown that

$$\begin{aligned} U(f, P'') &\leq U(f, P) \\ L(f, P') &\leq L(f, P'') \\ L(f, P'') &\leq U(f, P''). \end{aligned} \tag{3.23}$$

Together, these show that

$$U(f, P) \geq L(f, P'). \tag{3.24}$$

□

With this result in mind, we define the lower and upper Riemann integrals:

$$\begin{aligned} \underline{\int_Q} f &= \sup_P L(f, P) \\ \overline{\int_Q} f &= \inf_P U(f, P). \end{aligned} \tag{3.25}$$

Clearly, we have

$$\underline{\int_Q} f \leq \overline{\int_Q} f, \tag{3.26}$$

Finally, we define Riemann integrable.

Definition 3.6. A function f is *Riemann integrable over* Q if the lower and upper Riemann integrals coincide (are equal).

3.3 Conditions for Integrability

Our next problem is to determine under what conditions a function is (Riemann) integrable.

Let's look at a trivial case:

Claim. *Let* $F : Q \to \mathbb{R}$ *be the constant function* $f(x) = c$. *Then* f *is R. integrable over* Q, *and*

$$\int_Q c = cv(Q). \tag{3.27}$$

Proof. Let P be a partition, and let R be a rectangle belonging to P. We see that $m_R(f) = M_R(f) = c$, so

$$\begin{aligned} U(f, P) &= \sum_R M_R(f) v(R) = c \sum_R v(R) \\ &= cv(Q). \end{aligned} \tag{3.28}$$

Similarly,

$$L(f, P) = cv(Q). \tag{3.29}$$

□

Corollary 3. *Let Q be a rectangle, and let $\{Q_i : i = 1, \dots, N\}$ be a collection of rectangles covering Q. Then*

$$v(Q) \leq \sum v(Q_i). \tag{3.30}$$

Theorem 3.7. *If $f : Q \to \mathbb{R}$ is continuous, then f is R. integrable over Q.*

Proof. We begin with a definition

Definition 3.8. Given a partition P of Q, we define

$$\text{mesh width}(P) = \sup_R \text{width}(R). \tag{3.31}$$

Remember that

$$Q \text{ compact} \implies f : Q \to \mathbb{R} \text{ is uniformly continuous.} \tag{3.32}$$

That is, given $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in Q$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Choose a partition P of Q with mesh width less than δ. Then, for every rectangle R belonging to P and for every $x, y \in R$, we have $|x - y| < \delta$. By uniform continuity we have, $M_R(f) - m_R(f) \leq \epsilon$, which is used to show that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_R (M_R(f) - m_R(f))v(R) \\ &\leq \epsilon \sum v(R) \\ &\leq \epsilon v(Q). \end{aligned} \tag{3.33}$$

We can take $\epsilon \to 0$, so

$$\sup_P L(f, P) = \inf_P U(f, P), \tag{3.34}$$

which shows that f is integrable. □

We have shown that continuity is sufficient for integrability. However, continuity is clearly not necessary. What is the general condition for integrability? To state the answer, we need the notion of *measure zero.*

Definition 3.9. Suppose $A \subseteq \mathbb{R}^n$. The set A is of *measure zero* if for every $\epsilon > 0$, there exists a countable covering of A by rectangles $Q_1, Q_2, Q_3, \dots$ such that $\sum_i v(Q_i) < \epsilon$.

Theorem 3.10. *Let $f : Q \to \mathbb{R}$ be a bounded function, and let $A \subseteq Q$ be the set of points where f is* not *continuous. Then f is R. integrable if and only if A is of measure zero.*

Before we prove this, we make some observations about sets of measure zero:

1. Let $A, B \subseteq \mathbb{R}^n$ and suppose $B \subset A$. If A is of measure zero, then B is also of measure zero.

2. Let $A_i \subseteq \mathbb{R}^n$ for $i = 1, 2, 3, \ldots$, and suppose the A_i's are of measure zero. Then $\cup A_i$ is also of measure zero.

3. Rectangles are *not* of measure zero.

We prove the second observation:

For any $\epsilon > 0$, choose coverings $Q_{i,1}, Q_{i,2}, \ldots$ of A_i such that each covering has total volume less than $\epsilon/2^i$. Then $\{Q_{i,j}\}$ is a countable covering of $\cup A_i$ of total volume

$$\sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon. \tag{3.35}$$