Lecture 26

We continue our study of forms with compact support. Let us begin with a review. Let $U \in \mathbb{R}^n$ be open, and let

$$\omega = \sum_I f_I(x_1, \dots, x_n) dx_I, \tag{5.48}$$

where $I = (i_1, \dots, i_k)$ is strictly increasing and $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$. Then

$$\omega \text{ is compactly supported} \iff \text{every } f_I \text{ is compactly supported.} \tag{5.49}$$

By definition,

$$\text{supp } f_I = \overline{\{x \in U : f_I(x) \neq 0\}}. \tag{5.50}$$

We assume that the f_I's are $\mathcal{C}^2$ maps.

Notation.

$$\Omega^k_c(U) = \text{ space of compactly supported differentiable } k\text{-forms on } U. \tag{5.51}$$

Now, let $\omega \in \Omega^n_c(U)$ defined by

$$\omega = f(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n, \tag{5.52}$$

where $f \in \Omega^0_c(U)$. Then

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n. \tag{5.53}$$

Last time we proved the Poincare Lemma for open rectangles R in $\mathbb{R}^n$. We assumed that $\omega \in \Omega^n_c(\text{Int } R)$. That is, we assumed that $\omega \in \Omega^n_c(\mathbb{R}^n)$ such that supp $\omega \subset \text{Int } R$. We showed that for such ω the following two conditions are equivalent:

1. $\int_{\mathbb{R}^n} \omega = 0$,
2. There exists a $\mu \in \Omega^{n-1}_c(\text{Int } R)$ such that $d\mu = 0$.

Definition 5.9. Whenever $\omega \in \Omega^k(U)$ and $\omega = d\mu$ for some $\mu \in \Omega^{k-1}(U)$, we say that ω is *exact.*

Definition 5.10. Whenever $\omega \in \Omega^k(U)$ such that $d\omega = 0$, we say that ω is *closed.*

Observe that

$$\omega \in \Omega^n_c(U) \implies d\omega = 0. \tag{5.54}$$

Now we prove the Poincare Lemma for open connected subsets of $\mathbb{R}^n$.

Poincare Lemma. *Let U be a connected open subset of $\mathbb{R}^n$, and let $\omega \in \Omega^n_c(U)$. The following conditions are equivalent:*

1. $\int_U \omega = 0$,
2. $\omega = d\mu$, *for some* $\mu \in \Omega^{n-1}_c(U)$.

Proof. We prove this more general case by reducing the proof to the case where U is a rectangle, which we proved in the previous lecture.

First we prove that (2) implies (1). We can choose a family of rectangles $\{R_i, i \in \mathbb{N}\}$ such that

$$U = \bigcup_{i\in\mathbb{N}} \text{Int } R_i \tag{5.55}$$

Since the support of μ is compact, the set supp μ is covered by finitely many of the rectangles.

We take a partition of unity $\{\phi_i, i \in \mathbb{N}\}$ subordinate to $\{R_i\}$, so that

$$\mu = \sum_{i=1}^{N} \underbrace{\phi_i\mu}) \text{supported on Int } R_i \tag{5.56}$$

Then

$$\int d\mu = \sum_i \int_{\mathbb{R}^n} d(\phi_i\mu). \tag{5.57}$$

Each term on the r.h.s is zero by the Poincare Lemma we proved last lecture.

We now prove the other direction, that (1) implies (2). It is equivalent to show that if $\omega_1, \omega_2 \in \Omega^n_c(U)$ such that

$$\int \omega_1 = \int \omega_2, \tag{5.58}$$

then $\omega_1 \sim \omega_2$, meaning that there exists a form $\mu \in \Omega^{n-1}_c(U)$ such that $\omega_1 = \omega_2 + d\mu$.

Choose a partition of unity $\{\phi_i\}$ as before. Then

$$\omega = \sum_{i=1}^{M} \underbrace{\phi_i\omega}_{\text{supported on Int } R_i} \tag{5.59}$$

Let

$$\int \omega = c \in \mathbb{R}, \tag{5.60}$$

and

$$\int \phi_i\omega = c_i. \tag{5.61}$$

Choose a form ω_0 such that

$$\int \omega_0 = 1 \tag{5.62}$$

and such that $\text{supp } \omega_0 \subseteq Q_0 = R_j$ for some j. Then

$$\int \underbrace{\phi_i \omega}_{\text{supported in } R_i} = \int \underbrace{c_i \omega_0}_{\text{supported in } Q_0} \tag{5.63}$$

We want to show that there exists $\mu_i \in \Omega_c^{n-1}(U)$ such that $\phi_i \omega = c_i \omega_i + d\mu_i$.

Now we use the fact that U is connected. We use the following lemma.

Lemma 5.11. *Let U be connected. Given rectangles R_i such that* $\text{supp } \phi_i \omega \subset \text{Int } R_i$, *and given a fixed rectangle Q_0 and any point $x \in U$, there exists a finite sequence of rectangles $R_0, \ldots, R_N$ with the following properties: $Q_0 = R_0$, $x \in \text{Int } R_N$, and* $(\text{Int } R_i) \cap (\text{Int } R_{i+1})$ *is non-empty.*

We omit the proof of this lemma.

Now, define $\omega_i = \phi_i \omega$, so

$$\int \omega_i = \int c_i \omega_0. \tag{5.64}$$

Note that

$$\text{supp } (c_i \omega_0) \subseteq \text{Int } (Q_0) \tag{5.65}$$

$$\text{supp } (\omega_i) \subseteq \text{Int } (R_i). \tag{5.66}$$

Choose forms ν_i such that $\text{supp } \nu_i \subseteq \text{Int } R_i \cap \text{Int } R_{i+1}$ and such that

$$\int \nu_i = 1. \tag{5.67}$$

This implies that

$$\text{supp } (\nu_i - \nu_{i+1}) \subseteq \text{Int } R_{i+1} \tag{5.68}$$

By definition,

$$\int (\nu_i - \nu_{i+1}) = 0. \tag{5.69}$$

By the Poincare Lemma we proved last time, $\nu_i \sim \nu_{i+1}$, so there exists $\mu_i \in \Omega_c^{n-1}(U)$ such that $\nu_i = \nu_{i+1} + d\mu_i$.

So,

$$c_i \omega_0 \sim c_i \nu_0 \sim c_i \nu_1 \sim \ldots \sim c_i \nu_N \sim \phi_i \omega. \tag{5.70}$$

□

5.2 Proper Maps and Degree

We introduce a class of functions that remain compactly supported under the pullback operation.

Definition 5.12. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^k$, and let $f : U \to V$ be a continuous map. The map f is *proper* if for all compact subsets $K \subseteq V$, the set $f^{-1}(K)$ is compact.

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^k$, and let $f : U \to V$ be a continuous map. Also let $\omega \in \Omega^k(V)$. The map

$$f^* : \Omega^k(V) \to \Omega^k(U) \tag{5.71}$$

is defined such that

$$\omega = g(y_1, \dots, y_n) dy_{i_1} \wedge \dots \wedge dy_{i_k} \to f^*\omega = g(f(x)) df_{i_1} \wedge \cdots df_{i_k}. \tag{5.72}$$

So,

$$f^{-1}(\text{supp } \omega) \supseteq \text{supp}\,(f^*\omega). \tag{5.73}$$

If f is proper and $\omega \in \Omega^n_c(V)$, then $\text{supp}\,(f^*\omega)$ is compact, in which case the map f^* is actually of the form

$$f^* : \Omega^k_c(V) \to \Omega^k_c(U). \tag{5.74}$$

That is, $\omega \in \Omega^n_c(V) \to f^*\omega \in \Omega^n_c(U)$. So, it makes sense to take the integral

$$\int_U f^*\omega = (\deg f) \int_V \omega. \tag{5.75}$$

Theorem 5.13. *Let U, V be connected open subsets of $\mathbb{R}^n$, and let $f : U \to V$ be a $\mathcal{C}^\infty$ map. For all $\omega \in \Omega^n_c(V)$,*

$$\int_U f^*\omega = (\deg f) \int_V \omega. \tag{5.76}$$

Proof. Take $\omega_0 \in \Omega^n_c(V)$ such that

$$\int \omega_0 = 1. \tag{5.77}$$

Define

$$\deg f \equiv \int f^*\omega_0, \tag{5.78}$$

and suppose that

$$\int \omega = c. \tag{5.79}$$

Then

$$\int \omega = \int c\omega_0. \tag{5.80}$$

By the Poincare Lemma, $\omega \sim c\omega_0$. That is, there exists $\mu \in \Omega_c^{n-1}(V)$ such that $\omega = c\omega_0 + d\mu$. Then

$$\begin{aligned} f^*\omega &= f^*(c\omega_0) + f^*(d\mu) \\ &= f^*(c\omega_0) + d(f^*\mu), \end{aligned} \tag{5.81}$$

which shows that $f^*\omega \sim f^*(c\omega_0)$. Putting this altogether,

$$\begin{aligned} \int f^*\omega &= \int f^*(c\omega_0) \\ &= c\int f^*\omega_0 \\ &= c \deg f \\ &= \left(\int \omega\right) \deg f. \end{aligned} \tag{5.82}$$

□

We had $\omega = g(y_1, \dots, y_n) dy_1 \wedge \cdots \wedge dy_n$, so

$$\begin{aligned} f^*\omega &= g(f(x)) df_1 \wedge \cdots \wedge df_m \\ &= g(f(x)) \det \left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n, \end{aligned} \tag{5.83}$$

where we used the fact that

$$df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j \tag{5.84}$$

Restated in coordinates, the above theorem says that

$$\begin{aligned} &\int_U g(f(x)) \det(Df) dx_1 \wedge \cdots \wedge dx_n \\ &\qquad = (\deg f) \int_V g(y_1, \dots, y_n) dy_1 \wedge \cdots \wedge dy_n. \end{aligned} \tag{5.85}$$

Claim. *Given proper maps $f : V \to W$ and $g : U \to V$, where U, V, W are connected open subsets of $\mathbb{R}^n$,*

$$\deg(fg) = (\deg g)(\deg f). \tag{5.86}$$

Proof. Note that $(f \circ g)^* = g^* \circ f^*$, so

$$\begin{aligned} \deg(f \circ g) \int_W \omega &= \int_U (f \circ g)^* \omega \\ &= \int_V g^*(f^*\omega) \\ &= (\deg g) \int_V f^*\omega \\ &= (\deg g)(\deg f) \int_W \omega. \end{aligned} \tag{5.87}$$