Lecture 25

5.1 The Poincare Lemma

Let U be an open subset of $\mathbb{R}^n$, and let $\omega \in \Omega^k(U)$ be a k-form. We can write $\omega = \sum a_I dx_I, I = (i_1, \dots, i_k)$, where each $a_I \in \mathcal{C}^\infty(U)$. Note that

$$\omega \in \Omega_c^k \iff a_I \in \mathcal{C}_0^\infty(U) \text{ for each } I. \tag{5.17}$$

We are interested in $\omega \in \Omega_c^n(U)$, which are of the form

$$\omega = f dx_1 \wedge \cdots \wedge dx_n, \tag{5.18}$$

where $f \in \mathcal{C}_0^\infty(U)$. We define

$$\int_U \omega = \int_U f = \int_U f dx, \tag{5.19}$$

the Riemann integral of f over U.

Our goal over the next couple lectures is to prove the following fundamental theorem known as the Poincare Lemma.

Poincare Lemma. *Let U be a connected open subset of $\mathbb{R}^n$, and let $\omega \in \Omega_c^n(U)$. The following conditions are equivalent:*

1. $\int_U \omega = 0$,
2. $\omega = d\mu$, *for some* $\mu \in \Omega_c^{n-1}(U)$.

In today's lecture, we prove this for $U = \text{Int } Q$, where $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a rectangle.

Proof. First we show that (2) implies (1).

Notation.

$$dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \equiv dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n. \tag{5.20}$$

Let $\mu \in \Omega_c^{n-1}(U)$. Specifically, define

$$\mu = \sum_i f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \tag{5.21}$$

where each $f_i \in \mathcal{C}_0^\infty(U)$. Every $\mu \in \Omega_c^{n-1}(U)$ can be written this way.

Applying d we obtain

$$d\mu = \sum_i \sum_j \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n. \tag{5.22}$$

Notice that if $i \neq j$, then the i, jth summand is zero, so

$$\begin{aligned} d\mu &= \sum_i \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \sum (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n. \end{aligned} \tag{5.23}$$

Integrate to obtain

$$\int_U d\mu = \sum (-1)^{i-1} \int_U \frac{\partial f_i}{\partial x_i}. \tag{5.24}$$

Note that

$$\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i} dx_i = f_i(x)|_{x_i=a_i}^{x_i=b_i} = 0 - 0 = 0, \tag{5.25}$$

because f is compactly supported in U. It follows from the Fubini Theorem that

$$\int_U \frac{\partial f_i}{\partial x_i} = 0. \tag{5.26}$$

Now we prove the other direction, that (1) implies (2). Before our proof we make some remarks about functions of one variable.

Suppose $I = (a, b) \subseteq \mathbb{R}$, and let $g \in \mathcal{C}_0^\infty(I)$ such that supp $g \subseteq [c, d]$, where $a < c < d < b$. Also assume that

$$\int_a^b g(s) ds = 0. \tag{5.27}$$

Define

$$h(x) = \int_a^x g(s) ds, \tag{5.28}$$

where $a \leq x \leq b$.

Claim. *The function h is also supported on c, d.*

Proof. If $x > d$, then we can write

$$h(x) = \int_a^b g(s) ds - \int_x^b g(s) ds, \tag{5.29}$$

where the first integral is zero by assumption, and the second integral is zero because the integrand is zero. □

Now we begin our proof that (1) implies (2).

Let $\omega \in \Omega^n(U)$, where $U = Q$, and assume that

$$\int_U \omega = 0. \tag{5.30}$$

We will use the following inductive lemma:

Lemma 5.8. *For all $0 \leq k \leq n+1$, there exists $\mu \in \Omega_c^{n-1}(U)$ and $f \in \mathcal{C}_0^\infty(U)$ such that*

$$\omega = d\mu + f dx_1 \wedge \cdots \wedge dx_n \tag{5.31}$$

and

$$\int f(x_1, \ldots, x_n) dx_k \ldots dx_n = 0. \tag{5.32}$$

Note that the hypothesis for $k = 0$ and $\mu = 0$ says that $\int \omega = 0$, which is our assumption (1). Also note that the hypothesis for $k = n+1$, $f = 0$, and $\omega = d\mu$ is the same as the statement (2). So, if we can show that (the lemma is true for k) implies (the lemma is true for $k+1$), then we will have shown that (1) implies (2) in Poincare's Lemma. We now show this.

Assume that the lemma is true for k. That is, we have

$$\omega = d\mu + f dx_1 \wedge \cdots \wedge dx_n \tag{5.33}$$

and

$$\int f(x_1, \ldots, x_n) dx_k \ldots dx_n = 0, \tag{5.34}$$

where $\mu \in \Omega_c^{n-1}(U)$, and $f \in \mathcal{C}_0^\infty(\mathbb{R})$.

We can assume that μ and f are supported on Int Q', where $Q' \subseteq$ Int Q and $Q' = [c_1, d_1] \times \cdots \times [c_n, d_n]$.

Define

$$g(x_1, \ldots, x_k) = \int f(x_1, \ldots, x_n) d_{k+1} \ldots dx_n. \tag{5.35}$$

Note that g is supported on the interior of $[c_1, d_1] \times \cdots \times [c_k, d_k]$. Also note that

$$\int_{a_k}^{b_k} g(x_1, \ldots, x_{k-1}, s) ds = \int f(x_1, \ldots, x_n) dx_k \ldots dx_n = 0 \tag{5.36}$$

by our assumption that the lemma holds true for k.

Now, define

$$h(x_1, \ldots, x_k) = \int_{a_k}^{x_k} g(x_1, \ldots, x_{k-1}, s) ds. \tag{5.37}$$

From our earlier remark about functions of one variable, h is supported on $c_k \leq x_k \leq d_k$. Also, note that h is supported on $c_i \leq x_i \leq d_i$, for $1 \leq i \leq k-1$. We conclude therefore that h is also supported on $[c_1, d_1] \times \cdots \times [c_k, d_k]$.

Both g and its "anti-derivative" are supported.

$$\frac{\partial h}{\partial x_k} = g. \tag{5.38}$$

Let $\ell = n - k$, and consider $\rho = \rho(x_{k+1}, \ldots, x_n) \in \mathcal{C}_0^\infty(\mathbb{R}^\ell)$. Assume that ρ is supported on the rectangle $[c_{k+1}, d_{k+1}] \times \cdots \times [c_n, d_n]$ and that

$$\int \rho dx_{k+1} \ldots dx_n = 1. \tag{5.39}$$

We can always find such a function, so we just fix one such function.

Define

$$\nu = (-1)^k h(x_1, \ldots, x_k)\rho(x_{k+1}, \ldots, x_n)dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n. \tag{5.40}$$

The form ν is supported on $Q' = [c_1, d_1] \times \cdots \times [c_n, d_n]$.

Now we compute $d\nu$,

$$d\nu = (-1)^k \sum_j \frac{\partial}{\partial x_j}(h\rho)dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n. \tag{5.41}$$

Note that if $j \neq k$, then the summand is zero, so

$$\begin{aligned} d\nu &= (-1)^k \frac{\partial h}{\partial x_k}\rho dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n \\ &= (-1)\frac{\partial h}{\partial x_k}\rho dx_1 \wedge \cdots \wedge dx_n \\ &= -g\rho dx_1 \wedge \cdots \wedge dx_n. \end{aligned} \tag{5.42}$$

Now, define

$$\mu_{new} = \mu - \nu, \tag{5.43}$$

and

$$f_{new} = f(x_1, \ldots, x_n) - g(x_1, \ldots, x_k)\rho(x_{k+1}, \ldots, x_n). \tag{5.44}$$

$$\begin{aligned} \omega &= d\mu_{new} + f_{new}dx_1 \wedge \cdots \wedge dx_n \\ &= d\mu + (g(x_1, \ldots, x_k)\rho(x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_k) - g\rho)dx_1 \wedge \cdots \wedge dx_n \\ &= d\mu + f dx_1 \wedge \cdots \wedge dx_n \\ &= \omega. \end{aligned} \tag{5.45}$$

Note that

$$\begin{aligned} \int f_{new} &= \int f_{new}(x_1, \ldots, x_n)dx_{k+1} \ldots dx_n \\ &= \int f(x_1, \ldots, x_n)dx_{k+1} \ldots dx_n \\ &\qquad - g(x_1, \ldots, x_k) \int \rho(x_{k+1}, \ldots, x_n)dx_{k+1} \ldots dx_n \\ &= g(x_1, \ldots, x_k) - g(x_1, \ldots, x_k) = 0, \end{aligned} \tag{5.46}$$

which implies that the lemma is true for $k + 1$. □

Remark. In the above proof, we implicitly assumed that if $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then

$$g(x_1, \ldots, x_k) = \int f(x_1, \ldots, x_n)dx_{k+1} \ldots dx_m \tag{5.47}$$

is in $\mathcal{C}_0^\infty(\mathbb{R}^k)$. We checked the support, but we did not check that g is in $\mathcal{C}^\infty(\mathbb{R}^k)$. The proof of this is in the Supplementary Notes.